NUMERICAL METHODS FOR ULTRAPARABOLIC EQUATIONS

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Abstract. We analyze semidiscrete and fully discrete finite element approximations to the solution of an initial boundary value problem for a model ultraparabolic equation.

0. Introduction

In this paper we shall analyze semidiscrete as well as fully discrete finite element approximations to the solution of the following initial boundary value problem for a model ultraparabolic equation: Let $S, T > 0$, and $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary $\partial\Omega$, and let $\lambda = \lambda(x,s,t)$ be a positive smooth function on $\bar{\Omega} \times [0,S] \times [0,T]$. We seek a function $u : \bar{\Omega} \times [0,S] \times [0,T] \to \mathbb{R}$ satisfying

$$\begin{cases} u_t + \lambda u_s - \Delta u = 0, & \text{in } \Omega \times [0,S] \times [0,T], \\ u = 0, & \text{on } \partial\Omega \times [0,S] \times [0,T], \\ u(\cdot,0,\cdot) = v^0, \ \text{on } \bar{\Omega} \times [0,T], & u(\cdot,\cdot,0) = w^0, \ \text{on } \bar{\Omega} \times [0,S]. \end{cases} \tag{0.1}$$

In the sequel we shall think of both variables s and t as "time" variables, and v^0 and w^0 as initial data. We shall always assume below that the data of (0.1) are such that the problem possesses a unique solution which is sufficiently smooth for the approximation results that will be proved in the sequel. The operator $-\Delta$ is chosen for simplicity only; it could be replaced in what follows by any elliptic second order operator, with coefficients depending smoothly on x, s, and t.

Taking inner products of both sides of the differential equation in (0.1) by u, and using Green's formula, we obtain, with $\kappa = \sqrt{\lambda}$,

$$\frac{\partial}{\partial t}\|u\|^2 + \frac{\partial}{\partial s}\|\kappa u\|^2 + \|\nabla u\|^2 = (\lambda_s u, u) \le C\|u\|^2, \quad \text{with } \|\cdot\| = \|\cdot\|_{L_2(\Omega)},$$

whence

$$\frac{\partial}{\partial t}(e^{-Ct}\|u\|^2) + \frac{\partial}{\partial s}(e^{-Ct}\|\kappa u\|^2) \le 0.$$

Let Γ be a piecewise smooth curve contained in $[0,S] \times [0,T]$ and connecting the intervals $[0,S]$ with $[0,T]$ on the s and t axes, and such that its normal has nonnegative components. Integrating in s and t we obtain by Green's formula, with C independent

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of Γ,

$$|||u|||_\Gamma^2 = \int_\Gamma \|u\|^2\,d\sigma \le C\big(\int_0^T \|v^0(t)\|^2\,dt + \int_0^S \|w^0(s)\|^2\,ds\big),$$

which thus shows the stability and uniqueness of the solution of (0.1) with respect to initial data. (Here and often below we suppress the dependence on x in the notation.)

We remark that in the particular case that λ is independent of x, the differential equation reduces to a family of standard parabolic equations along the characteristics of the equation $u_t + \lambda u_s = 0$. In this case we have

$$\big(\frac{\partial}{\partial t} + \lambda\frac{\partial}{\partial s}\big)\|u\|^2 + \|\nabla u\|^2 = 0.$$

In particular, $\|u\|$ is nonincreasing along the characteristics, and the estimate

$$\sup_{[0,S]\times[0,T]} \|u\| \le \max\big(\sup_{[0,T]}\|v^0\|, \sup_{[0,S]}\|w^0\|\big) \tag{0.2}$$

follows.

From the parabolic character of the equation along characteristics it follows in this case that the solution is smooth in x and in the variable along characteristics for $s, t \ne 0$, even without regularity assumptions on the initial data, but no regularity could be expected in the transversal variable without such regularity assumptions.

The maximum-norm estimate

$$\sup_{\bar\Omega\times[0,S]\times[0,T]} |u| \le \max\big(\sup_{\bar\Omega\times[0,T]} |v^0|, \sup_{\bar\Omega\times[0,S]} |w^0|\big)$$

also holds, and can be shown in a similar way as in the proof of the maximum principle for a standard parabolic equation, even in the case when λ depends on x, s, and t, but it is not clear that an estimate such as (0.2) holds if λ depends on x.

Ultraparabolic equations have several applications, for instance in probability, in the theory of Brownian motion, and in the theory of boundary layers, cf., e.g., Kolmogorov [3], [4], Genčev [1] and references therein. For existence and uniqueness results, other properties of ultraparabolic equations, and further references we refer the reader to Genčev [1], Il'in [2], Vladimirov and Drožžinov [7], and Tersenov [5].

The plan of the paper is as follows: In Section 1 we treat discretization with respect to the spatial variables only, using standard finite elements of order r. We show an optimal order estimate with respect to the norm $|||\cdot|||_\Gamma$ introduced above, and remark that a corresponding maximum-norm error estimate holds when λ is independent of x. In Section 2 we consider a fully discrete implicit backward Euler method, using backward difference quotients to approximate the derivatives with respect to both s and t in (0.1), and show fully discrete analogues of the above error estimates in the semidiscrete case. In Section 3, finally, we discuss a second order fully discrete scheme based on a variant of the box scheme for the hyperbolic part of the equation and show a corresponding optimal order error estimate.

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1. Semidiscretization in space

Let $\{S_h\}$ represent a family of finite-dimensional subspaces of $H_0^1(\Omega)$. Such spaces typically consist of piecewise polynomial functions of degree at most $r-1$ defined on suitable partitions of Ω, where $r \geq 2$ is an integer. We assume that these spaces are such that there is a constant C independent of h such that for each $v \in H^q(\Omega) \cap H_0^1(\Omega)$, $2 \leq q \leq r$, there exists $\chi \in S_h$ with

$$\|v-\chi\| + h\|\nabla(v-\chi)\| \leq Ch^q\|v\|_{H^q(\Omega)}. \tag{1.1}$$

We define the spatially semidiscrete approximation $u_h = u_h(\cdot, s, t) \in S_h,\ (s,t) \in [0,S]\times[0,T]$, of the solution of (0.1) by

$$\begin{aligned} &(u_{ht}, \chi) + (\lambda u_{hs}, \chi) + (\nabla u_h, \nabla\chi) = 0, \quad \forall \chi \in S_h,\ (s,t) \in [0,S]\times[0,T], \\ &u_h(\cdot,0,\cdot) = v_h^0, \text{ in } \Omega\times[0,T], \quad u_h(\cdot,\cdot,0) = w_h^0, \text{ in } \Omega\times[0,S], \end{aligned} \tag{1.2}$$

where v_h^0, w_h^0 are such that $v_h^0(0) = w_h^0(0)$ and

$$\|v^0(t) - v_h^0(t)\| + \|w^0(s) - w_h^0(s)\| \leq Ch^r, \quad \text{for } (s,t) \in [0,S]\times[0,T]. \tag{1.3}$$

In terms of a basis for S_h the semidiscrete problem (1.2) can be formulated as an initial boundary value problem for a first order symmetric hyperbolic system of Friedrichs' type. Thus, in particular, the existence of the semidiscrete approximation u_h follows easily from standard theory for such systems.

For the purpose of error estimation we will use the elliptic (or Ritz) projection $R_h : H_0^1(\Omega) \to S_h$, commonly used in the error analysis for parabolic equations, defined by

$$(\nabla R_h v, \nabla\chi) = (\nabla v, \nabla \chi) \quad \forall v \in H_0^1(\Omega) \quad \forall \chi \in S_h. \tag{1.4}$$

It is well known from the error analysis for elliptic problems that, under the assumption (1.1), we have

$$\|R_h v - v\| + h\|\nabla(R_h v - v)\| \leq Ch^r\|v\|_{H^r(\Omega)}, \quad \forall v \in H^r(\Omega) \cap H_0^1(\Omega), \tag{1.5}$$

with a constant C independent of v and h.

Theorem 1. *Let u_h and u be the solutions of* (1.2) *and* (0.1), *respectively, and assume that* (1.3) *holds. Then, for any curve Γ as in Section 0,*

$$|||u_h - u|||_\Gamma \leq C(u)h^r.$$

Proof. We set $W = R_h u$ and write the error $u_h - u = \rho + \theta$, where $\rho = W - u$ and $\theta = u_h - W$. By the elliptic error estimate (1.5) we have at once

$$\|\rho(s,t)\| \leq Ch^r, \quad \text{for } (s,t) \in [0,S]\times[0,T]. \tag{1.6}$$

In order to estimate θ, we first note that for the initial-values

$$\|\theta(0,t)\| + \|\theta(s,0)\| = \|v_h^0(t) - R_h v^0(t)\| + \|w_h^0(s) - R_h w^0(s)\| \leq Ch^r. \tag{1.7}$$

Further, by (1.4) and (0.1),

$$\begin{aligned}(\theta_t,\chi)+(\lambda\theta_s,\chi)+(\nabla\theta,\nabla\chi)&=(W_t,\chi)+(\lambda W_s,\chi)+(\nabla W,\nabla\chi)\\&=(W_t,\chi)+(\lambda W_s,\chi)-(\Delta u,\chi),\quad\forall\chi\in S_h,\end{aligned}$$

i.e.,

$$(\theta_t,\chi)+(\lambda\theta_s,\chi)+(\nabla\theta,\nabla\chi)=-(\rho_t,\chi)-(\lambda\rho_s,\chi),\quad\forall\chi\in S_h.$$

Since R_h commutes with differentiation with respect to s and to t the estimate (1.6) holds also for ρ_s and ρ_t. Therefore, choosing $\chi=\theta(\cdot,s,t)$ and noting that $2(\lambda\theta_s,\theta)=(\lambda\theta,\theta)_s-(\lambda_s\theta,\theta)$, we obtain by obvious estimates, with $\kappa=\sqrt{\lambda}$,

$$\frac{\partial}{\partial t}\|\theta\|^2+\frac{\partial}{\partial s}\|\kappa\theta\|^2\le C\|\theta\|^2+Ch^{2r}.$$

Multiplying by e^{-Ct}, integrating in s and t, and using Green's formula, we obtain

$$|||\theta|||_\Gamma^2=\int_\Gamma\|\theta\|^2\,d\sigma\le Ch^{2r}.$$

Since by (1.6), $|||\rho|||_\Gamma\le Ch^r$, this completes the proof. □

We note that as in Section 0, in the case when λ is independent of x, (1.2) represents a family of semidiscrete approximations of parabolic equations along the characteristics. In this case we may conclude from known results, by integrating along the characteristics, that

$$\sup_{[0,S]\times[0,T]}\|u_h-u\|\le C(u)h^r.$$

2. The backward Euler method

We shall now turn to discretization also in time and analyze a fully discrete scheme based on using the backward Euler method for the hyperbolic part of (0.1).

Letting k be a time step and $s^m=mk, t^n=nk, m,n=0,\dots,$ we shall denote the values of our functions on the grid by $V^{m,n}=V(\cdot,s^m,t^n)$. We shall also use the backward difference quotients

$$\bar\partial_1V^{m,n}=(V^{m,n}-V^{m-1,n})/k,\quad\bar\partial_2V^{m,n}=(V^{m,n}-V^{m,n-1})/k.$$

We now define the fully discrete backward Euler method for (0.1) by

$$\begin{aligned}&(\bar\partial_2U^{m,n},\chi)+(\lambda^{m,n}\bar\partial_1U^{m,n},\chi)+(\nabla U^{m,n},\nabla\chi)=0,\ \forall\chi\in S_h,\ m,n\ge1,\\&U^{0,n}=v_h^0(t^n),\quad U^{m,0}=w_h^0(s^m),\ m,n\ge0,\end{aligned}\tag{2.1}$$

where the initial approximations are chosen as in Section 1. It is readily seen that the approximations $U^{m,n}$ are well defined; indeed, (2.1) represents a $N\times N$ linear system for $U^{m,n}$ with symmetric and positive definite matrix, where $N=\dim S_h$.

In order to express our error estimate we shall need to define an analogue of the norms $|||\cdot|||_\Gamma$ used in Section 0. For this purpose, let $\Sigma_k\subset[0,S]\times[0,T]$ be a union of mesh-rectangles, which we take to mean a set of meshpoints in a rectangle of the form $(0, s^m] \times (0, t^n]$. Let $\mathit{\Gamma}_k$ be the discrete forward boundary of $\mathit{\Sigma}_k$, consisting of the meshpoints of $\mathit{\Sigma}_k$ such that not all four neighboring meshpoints belong to $\mathit{\Sigma}_k$, and set

$$|||V|||_{\mathit{\Gamma}_k} = \Big(k \sum_{(s^m,t^n)\in \mathit{\Gamma}_k} (V^{m,n})^2\Big)^{\frac{1}{2}}.$$

We then have the following.

Theorem 2. *Let $U^{m,n}$ and u be the solutions of* (2.1) *and* (0.1), *respectively, and assume that the initial approximations satisfy* (1.3). *Then, for $\mathit{\Gamma}_k$ as above,*

$$|||U - u|||_{\mathit{\Gamma}_k} \le C(u)(h^r + k).$$

Proof. We write again the error $U - u = (U - W) + (W - u) = \theta + \rho$ where $W = R_h u$, and recall the estimate (1.6) for ρ. In order to estimate θ, we first note that for the initial-values, as in (1.7), we have

$$\|\theta^{0,n}\| + \|\theta^{m,0}\| \le Ch^r. \tag{2.2}$$

Further, arguing as in the proof of Theorem 1

$$(\bar\partial_2\theta^{m,n}, \chi) + (\lambda^{m,n}\bar\partial_1\theta^{m,n}, \chi) + (\nabla\theta^{m,n}, \nabla\chi) = (\omega^{m,n}, \chi), \tag{2.3}$$

where $\omega^{m,n} = \omega_1^{m,n} + \omega_2^{m,n} + \omega_3^{m,n}$, with

$$\begin{aligned}
\omega_1^{m,n} &= -(R_h - I)\bar\partial_2 u^{m,n},\\
\omega_2^{m,n} &= -\lambda^{m,n}(R_h - I)\bar\partial_1 u^{m,n},\\
\omega_3^{m,n} &= -(\bar\partial_2 u^{m,n} - u_t(s^m, t^n)) - \lambda^{m,n}(\bar\partial_1 u^{m,n} - u_s(s^m, t^n)).
\end{aligned}$$

Now since

$$\omega_1^{m,n} = -\frac{1}{k}\int_{t^{n-1}}^{t^n} (R_h - I)u_t(s^m, \tau)\, d\tau,$$

and similarly for $\omega_2^{m,n}$, we easily see that

$$\|\omega_1^{m,n}\| + \|\omega_2^{m,n}\| \le Ch^r.$$

Together with the obvious $O(k)$ estimate for $\omega_3^{m,n}$ we conclude that

$$\|\omega^{m,n}\| \le C(h^r + k).$$

Choosing $\chi = \theta^{m,n}$ in (2.3) and using

$$2(\bar\partial_2\theta^{m,n}, \theta^{m,n}) = \bar\partial_2\|\theta^{m,n}\|^2 + k\|\bar\partial_2\theta^{m,n}\|^2 \ge \bar\partial_2\|\theta^{m,n}\|^2,$$

and for the second term the analogous estimate together with the differentiability of λ, we obtain, again with $\kappa = \sqrt{\lambda}$ and C independent of $\mathit{\Gamma}$,

$$\bar\partial_2\|\theta^{m,n}\|^2 + \bar\partial_1\|\kappa^{m,n}\theta^{m,n}\|^2 \le C\|\theta^{m-1,n}\|^2 + C(h^r + k)^2.$$

Using the fact that, with obvious one-dimensional notation,

$$\bar\partial(e^{-Ct^n}V^n) \le e^{-Ct^n}(\bar\partial V^n - CV^{n-1}), \quad \text{for } V^n \ge 0, \tag{2.4}$$

we obtain

$$\bar\partial_2(e^{-Ct^n}\|\theta^{m,n}\|^2) + \bar\partial_1(e^{-Ct^n}\|\kappa^{m,n}\theta^{m,n}\|^2) \le C(h^r + k)^2,$$

from which the result now follows by summation over Σ_k. □

We remark that, as in the semidiscrete case, when λ does not depend on x, we can improve the estimate of Theorem 2 to the discrete maximum-norm in time estimate

$$\max_{(s^m,t^n)\in[0,S]\times[0,T]} \|U^{m,n} - u^{m,n}\| \le C(u)(h^r + k).$$

Here this is not accomplished using the characteristics, but by the following energy argument: Choosing $\chi = \theta^{m,n}$ in (2.3) and using the Cauchy-Schwarz inequality we get easily

$$\bar\partial_2\|\theta^{m,n}\| + \lambda^{m,n}\bar\partial_1\|\theta^{m,n}\| \le \|\omega^{m,n}\| \le C(h^r + k).$$

Hence

$$\|\theta^{m,n}\| \le \max(\|\theta^{m-1,n}\|, \|\theta^{m,n-1}\|) + Ck(h^r + k),$$

and, by induction,

$$\|\theta^{m,n}\| \le \max(\max_{s^m\in[0,S]}\|\theta^{m,0}\|, \max_{t^n\in[0,T]}\|\theta^{0,n}\|) + C(s^m + t^n)(h^r + k).$$

Together with the usual estimates (1.6) for ρ and (1.7) for the initial-values of θ, this completes the proof.

3. The box scheme

In this final section we shall consider a second order fully discrete scheme based on the box scheme for the hyperbolic part of the equation (0.1).

In addition to our earlier notation we shall use

$$\begin{aligned} &V^{m-\frac12,n} = \frac12(V^{m,n} + V^{m-1,n}),\ V^{m,n-\frac12} = \frac12(V^{m,n} + V^{m,n-1}),\\ &\hat V^{m-\frac12,n-\frac12} = \frac12(V^{m,n} + V^{m-1,n-1}).\end{aligned}$$

With $\lambda_{m-\frac12,n-\frac12} = \lambda(\cdot, s^{m-\frac12}, t^{n-\frac12})$ (with lower index to distinguish from the corresponding average), we define $U^{m,n}$ by

$$\begin{aligned} &(\bar\partial_2 U^{m-\frac12,n}, \chi) + (\lambda_{m-\frac12,n-\frac12}\bar\partial_1 U^{m,n-\frac12}, \chi) + (\nabla \hat U^{m-\frac12,n-\frac12}, \nabla\chi) = 0,\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall \chi\in S_h,\ m,n\ge 1,\\ &U^{0,n} = v_h^0(t^n),\quad U^{m,0} = w_h^0(s^m),\ m,n\ge 0,\end{aligned} \tag{3.1}$$

where the initial approximations are chosen as in Section 1.

We shall prove the following error estimate where $|\!|\!|\cdot|\!|\!|_{\Gamma_k}$ is defined as in Section 2. Our analysis is based on [6]. We remark that a more standard way to define the box scheme would have been to use instead of $\hat U^{m-\frac12,n-\frac12}$ the four point average $U^{m-\frac12,n-\frac12} = (U^{m,n} + U^{m-1,n} + U^{m,n-1} + U^{m-1,n-1})/4$. The analysis could be pursued also with this choice but would lead to error estimates at the midpoints of the edges of the mesh-boxes. We shall not insist on the details.

Theorem 3. *Let $U^{m,n}$ and u be the solutions of* (3.1) *and* (0.1)*, respectively, and assume that the initial approximations satisfy* (1.3)*. Then, for $\mathit{\Gamma}_k$ as above,*

$$|\!|\!|U - u|\!|\!|_{\mathit{\Gamma}_k} \le C(u)(h^r + k^2).$$

Proof. Let as above $\rho = W - u, \theta = U - W$. Then θ satisfies

$$(\bar\partial_2 \theta^{m-\frac12,n}, \chi) + (\lambda_{m-\frac12,n-\frac12}\bar\partial_1\theta^{m,n-\frac12}, \chi) + (\nabla\hat\theta^{m-\frac12,n-\frac12}, \nabla\chi) = (\omega^{m,n}, \chi), \tag{3.2}$$

where $\omega^{m,n} = \sum_{j=1}^4 \omega_j^{m,n}$, with

$$\begin{aligned}
\omega_1^{m,n} &= -(R_h - I)\bar\partial_2 u^{m-\frac12,n},\\
\omega_2^{m,n} &= -\lambda_{m-\frac12,n-\frac12}(R_h - I)\bar\partial_1 u^{m,n-\frac12},\\
\omega_3^{m,n} &= u_t(s^{m-\frac12}, t^{n-\frac12}) - \bar\partial_2 u^{m-\frac12,n}\\
&\qquad\qquad + \lambda_{m-\frac12,n-\frac12}(u_s(s^{m-\frac12}, t^{n-\frac12}) - \bar\partial_1 u^{m,n-\frac12}),\\
\omega_4^{m,n} &= \mathit{\Delta}(\hat u^{m-\frac12,n-\frac12} - u(s^{m-\frac12}, t^{n-\frac12})).
\end{aligned}$$

In view of (1.5) we have

$$\|\omega^{m,n}\| \le C(h^r + k^2). \tag{3.3}$$

Choosing $\chi = \hat\theta^{m-\frac12,n-\frac12}$ in (3.2) and using (3.3) we easily obtain

$$\begin{aligned}
&(\bar\partial_2\theta^{m-\frac12,n}, \hat\theta^{m-\frac12,n-\frac12}) + (\lambda_{m-\frac12,n-\frac12}\bar\partial_1\theta^{m,n-\frac12}, \hat\theta^{m-\frac12,n-\frac12}) + \|\nabla\hat\theta^{m-\frac12,n-\frac12}\|^2\\
&\le \epsilon\|\hat\theta^{m-\frac12,n-\frac12}\|^2 + C(h^r + k^2)^2 \le \|\nabla\hat\theta^{m-\frac12,n-\frac12}\|^2 + C(h^r + k^2)^2,
\end{aligned}$$

for ϵ suitably chosen.

Using the identity

$$2(a + b - c - d)(b + c) = ((a + b)^2 - (c + d)^2) + ((b - d)^2 - (a - c)^2),$$

with a, b, c, d the values of θ at the four corners of the box $[s^{m-1}, s^m] \times [t^{n-1}, t^n]$, we find,

$$8(\bar\partial_2\theta^{m-\frac12,n}, \hat\theta^{m-\frac12,n-\frac12}) = 4\bar\partial_2\|\theta^{m-\frac12,n}\|^2 + k^2\bar\partial_1\|\bar\partial_2\theta^{m,n}\|^2,$$

and

$$\begin{aligned}
&8(\lambda_{m-\frac12,n-\frac12}\bar\partial_1\theta^{m,n-\frac12}, \hat\theta^{m-\frac12,n-\frac12})\\
&\quad= \frac4k\big(\|\kappa_{m-\frac12,n-\frac12}\theta^{m,n-\frac12}\|^2 - \|\kappa_{m-\frac12,n-\frac12}\theta^{m-1,n-\frac12}\|^2\big)\\
&\qquad\qquad + k\big(\|\kappa_{m-\frac12,n-\frac12}\bar\partial_1\theta^{m,n}\|^2 - \|\kappa_{m-\frac12,n-\frac12}\bar\partial_1\theta^{m,n-1}\|^2\big)\\
&\quad= 4\bar\partial_1\|\kappa_{m-\frac12,n-\frac12}\theta^{m,n-\frac12}\|^2 + k^2\bar\partial_2\|\kappa_{m-\frac12,n-\frac12}\bar\partial_1\theta^{m,n}\|^2\\
&\qquad\qquad + O\big(\|\theta^{m-1,n-\frac12}\|^2 + k^2\|\bar\partial_1\theta^{m,n-1}\|^2\big).
\end{aligned}$$

Letting

$$\begin{aligned}
P^{m,n} &= 4\|\theta^{m-\frac12,n}\|^2 + k^2\|\kappa_{m-\frac12,n-\frac12}\bar\partial_1\theta^{m,n}\|^2,\\
Q^{m,n} &= 4\|\kappa_{m-\frac12,n-\frac12}\theta^{m,n-\frac12}\|^2 + k^2\|\bar\partial_2\theta^{m,n}\|^2,
\end{aligned}$$

we have thus

$$\bar\partial_2 P^{m,n} + \bar\partial_1 Q^{m,n} \le C(h^r + k^2)^2 + C\big(Q^{m-1,n} + P^{m,n-1}\big).$$

As in the proof of Theorem 2 we obtain using (2.4) for both variables

$$\bar\partial_2(e^{-C(s^m+t^n)}P^{m,n}) + \bar\partial_1(e^{-C(s^m+t^n)}Q^{m,n}) \le C(h^r + k^2)^2.$$

Multiplying by k^2 and summing over Σ_k,

$$k \sum_{(s^m,t^n)\in\Gamma_k} \max\,(P^{m,n}, Q^{m,n}) \le k \sum_{s^m\in[0,S]} P^{m,0} + k \sum_{t^n\in[0,T]} Q^{0,n} + C(h^r + k^2)^2.$$

Note that $P^{m,n}$ and $Q^{m,n}$ are equivalent to $\|\theta^{m,n}\|^2 + \|\theta^{m-1,n}\|^2$ and $\|\theta^{m,n}\|^2 + \|\theta^{m,n-1}\|^2$, respectively. In view of (2.2) we conclude that

$$|\!|\!|\theta|\!|\!|^2_{\Gamma_k} \le C(h^r + k^2)^2.$$

Together with the estimate (1.6) for ρ this completes the proof of the theorem. □

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