STABILITY OF IMPLICIT AND IMPLICIT–EXPLICIT MULTISTEP METHODS FOR NONLINEAR PARABOLIC EQUATIONS

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Abstract. We analyze the discretization of nonlinear parabolic equations in Hilbert spaces by both implicit and implicit–explicit multistep methods and establish local stability under best possible and best possible *linear* stability conditions, respectively. Our approach is based on suitable quantifications of the non-self-adjointness of linear elliptic operators and a discrete perturbation argument.

1. Introduction

Let $T>0$ and $u^0\in H$, and consider the initial value problem, for a possibly nonlinear abstract parabolic equation,

$$\begin{cases} u'(t)+A(t)u(t)=B(t,u(t)), \quad 0<t<T,\\ u(0)=u^0, \end{cases} \tag{1.1}$$

in the usual triple of separable complex Hilbert spaces $V\subset H=H'\subset V'$, with V densely and continuously imbedded in H. Here $A(t):V\to V', t\in[0,T]$, is a linear operator, whereas the operator $B(t,\cdot):V\to V', t\in[0,T]$, may be nonlinear. Cf., e.g., [20, Chapter 3], [30, Chapters 3 and 4], [25, Sections 6.8.1, 6.9.4, 7.11.2, 9.3], [29, Chapter 4]. We assume that (1.1) possesses a unique, smooth solution.

Let (α,β) and (α,γ) be implicit and explicit q-step methods, respectively, generated by three polynomials α,β and γ,

$$\alpha(\zeta)=\sum_{i=0}^{q}\alpha_i\zeta^i,\quad \beta(\zeta)=\sum_{i=0}^{q}\beta_i\zeta^i,\quad \gamma(\zeta)=\sum_{i=0}^{q-1}\gamma_i\zeta^i,$$

with real coefficients α_i,β_i and γ_i.

Let $N\in\mathbb{N}, k:=T/N$ be the constant time step, and $t^n:=nk, n=0,\dots,N$, be a uniform partition of the interval $[0,T]$. Since we consider q-step schemes, we assume that starting approximations $U^0,\dots,U^{q-1}\in V$ are given. We recursively define a sequence of approximations $U^m\in V$ to the nodal values $u^m:=u(t^m)$ of the solution u of the initial value problem (1.1) by discretizing the differential equation in (1.1) either by the

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implicit scheme (α,β),

$$\sum_{i=0}^{q}\big(\alpha_i I + k\beta_i A(t^{n+i})\big)U^{n+i} = k\sum_{i=0}^{q}\beta_i B(t^{n+i},U^{n+i}), \tag{1.2}$$

$n=0,\dots,N-q$, with I the identity operator on H, or by the implicit–explicit scheme (α,β,γ),

$$\sum_{i=0}^{q}\big(\alpha_i I + k\beta_i A(t^{n+i})\big)U^{n+i} = k\sum_{i=0}^{q-1}\gamma_i B(t^{n+i},U^{n+i}), \tag{1.3}$$

$n=0,\dots,N-q$. The implicit–explicit scheme (α,β,γ) results by employing the implicit scheme (α,β) for the discretization of the linear part and the explicit scheme (α,γ) for the discretization of the nonlinear part of the differential equation; see [5, 6, 1]. The implicit scheme (1.2) has, in general, more advantageous stability properties than its implicit–explicit counterpart (1.3); however, if $B(t,\cdot)$ is nonlinear, (1.2) is a nonlinear equation in the unknown U^{n+q}. To approximate U^{n+q}, we need to linearize, for instance by the Newton method; we shall not study linearizations of (1.2) here. In contrast to (1.2), the unknown U^{n+q} appears in (1.3) only linearly, since $\gamma_q=0$; therefore, to advance with the implicit–explicit scheme (1.3) in time, we need to solve, at each time level, just one linear equation, which reduces to a linear system of equations, if we discretize also in space.

We assume that the implicit method (α,β) is strongly A(0)-stable and denote by $\vartheta, 0<\vartheta\le 90^\circ$, the largest angle for which the method (α,β) is A(ϑ)-stable.

1.1. **Abstract setting.** We denote by $(\cdot,\cdot)$ both the inner product on H and the antiduality pairing between V' and V, and by $|\cdot|$ and $\|\cdot\|$ the norms on H and V, respectively. The space V' may be considered the completion of H with respect to the dual norm $\|\cdot\|_\star$,

$$\|v\|_\star := \sup_{\substack{\tilde v\in V\\ \tilde v\ne 0}}\frac{|(v,\tilde v)|}{\|\tilde v\|} = \sup_{\substack{\tilde v\in V\\ \|\tilde v\|=1}}|(v,\tilde v)|.$$

We assume that the operator $A(t): V\to V'$ is uniformly *coercive* and *bounded*, i.e.,

$$\operatorname{Re}(A(t)v,v)\ge \kappa(t)\|v\|^2 \quad \forall v\in V \tag{1.4}$$

and

$$\|A(t)v\|_\star \le \nu(t)\|v\| \quad \forall v\in V, \tag{1.5}$$

respectively, with two positive functions, $\kappa,\nu:[0,T]\to\mathbb{R}$, uniformly bounded from below by a positive constant and from above, respectively. Operators satisfying (1.4) and (1.5) are sectorial in the sense that their numerical range $\{(A(t)v,v)/(v,v), v\in V, v\ne 0\}$ is contained in a sector $S_\varphi, S_\varphi := \{z\in\mathbb{C}: z=\rho e^{i\psi}, \rho\ge 0, |\psi|\le\varphi\}$, of half-angle $\varphi<90^\circ$.

High-order multistep schemes are not A-stable. To take advantage of the A(ϑ)-stability of the implicit scheme (α,β), in the case $\vartheta<90^\circ$, we need to quantify the non-self-adjointness of the operator $A(t)$. There are several equivalent measures of the non-self-adjointness of linear operators. Which measure or estimate of the non-self-adjointness of an operator is more suitable depends also on the employed stability technique. In the case of the energy technique, the ratio $\nu(t)/\kappa(t)$ is commonly used as an estimate of the non-self-adjointness of $A(t)$; notice, however, that this ratio depends on the norm $\|\cdot\|$ on V and may be a crude estimate of the non-self-adjointness of $A(t)$; see Section 4. On the other hand, the smallest half-angle of a sector containing the numerical range of $A(t)$ is a measure rather than an estimate of the non-self-adjointness of $A(t)$, which has been used in the stability analysis of multistep methods for *linear* parabolic equations by alternative stability techniques, such as Cauchy integral representations or spectral and Fourier techniques; see, e.g., [14, 15, 26]. As we shall see later on, sharp sufficient stability conditions for nonlinear parabolic equations are nonlinear in this measure of the non-self-adjointness of $A(t)$; see (4.14).

It will prove advantageous for our purposes to introduce alternative equivalent measures of the non-self-adjointness of $A(t)$. To this end, we first decompose $A(t)$ into its self-adjoint and anti-self-adjoint parts $A_s(t) := \big(A(t) + A(t)^\star\big)/2$ and $A_a(t) := \big(A(t) - A(t)^\star\big)/2$, respectively,

$$A(t) = A_s(t) + A_a(t), \quad t \in [0, T]. \tag{1.6}$$

Then, with the bounded linear operator $\mathcal{A}(t) : H \to H$ and its anti-self-adjoint part $\mathcal{A}_a(t)$,

$$\mathcal{A}(t) := A_s^{-1/2}(t)A(t)A_s^{-1/2}(t) = I + \mathcal{A}_a(t), \quad \mathcal{A}_a(t) = A_s^{-1/2}(t)A_a(t)A_s^{-1/2}(t), \tag{1.7}$$

with I the identity operator on H, for the operator $A(t)$ we have the equivalence

$$\forall v \in V \ \ (A(t)v, v) \in S_\varphi \iff |\mathcal{A}_a(t)| \le \tan\varphi \iff |\mathcal{A}(t)| \le \frac{1}{\cos\varphi}. \tag{1.8}$$

Indeed, for $v \in V$, with $\tilde v := A_s^{1/2}(t)v \in H$, we have

$$\begin{aligned}(A(t)v, v) &= (A(t)A_s^{-1/2}(t)\tilde v, A_s^{-1/2}(t)\tilde v) = (A_s^{-1/2}(t)A(t)A_s^{-1/2}(t)\tilde v, \tilde v)\\ &= (\mathcal{A}(t)\tilde v, \tilde v) = ((I + \mathcal{A}_a(t))\tilde v, \tilde v) = |\tilde v|^2 + (\mathcal{A}_a(t)\tilde v, \tilde v),\end{aligned}$$

i.e.,

$$(A(t)v, v) = |\tilde v|^2\Big[1 + \frac{(\mathcal{A}_a(t)\tilde v, \tilde v)}{|\tilde v|^2}\Big] \quad \forall v \in V, v \ne 0. \tag{1.9}$$

Due to the anti-self-adjointness of $\mathcal{A}_a(t)$, the second term in brackets is purely imaginary and (notice that $\mathrm{i}\mathcal{A}_a(t)$ is self-adjoint)

$$\sup_{\substack{\tilde v \in H\\ \tilde v \ne 0}} \frac{|(\mathcal{A}_a(t)\tilde v, \tilde v)|}{|\tilde v|^2} = |\mathcal{A}_a(t)|;$$

see, e.g., [27, Theorem 2.2.11]; the first equivalence in (1.8) follows immediately from the last relation and (1.9). Furthermore, it is easily seen that the norms of $\mathcal{A}_a(t)$ and $\mathcal{A}(t)$ are related by

$$|\mathcal{A}(t)|^2 = 1 + |\mathcal{A}_a(t)|^2 \tag{1.10}$$

(cf. (2.6) in the sequel); this yields the second equivalence in (1.8). In view of (1.8), the norms of both $\mathcal{A}_a(t)$ and $\mathcal{A}(t)$ are alternative measures of the non-self-adjointness of the linear operator $A(t)$.

Obviously, any stability condition on $|\mathcal{A}_a(t)|$ can be equivalently written as a stability condition on $|\mathcal{A}(t)|$ or on the smallest half-angle $\varphi(t)$ of the sector $S_{\varphi(t)}$ containing the numerical range of $A(t)$, and *vice versa*. Usually, stability analyses lead directly to *linear* sufficient stability conditions; indirectly, one may reformulate such conditions as nonlinear sufficient stability conditions on other quantities. It turns out that the best possible sufficient stability condition for the implicit scheme (1.2), under which we shall indeed establish stability, is linear in $|\mathcal{A}_a(t)|$ (and in the Lipschitz bound $\lambda_2(t)$ of $B(t,\cdot)$, see (1.12) in the sequel) and nonlinear in $|\mathcal{A}(t)|$ or $\varphi(t)$. Thus, our crucial assumption on the linear operator $A(t)$ is

$$|\mathcal{A}_a(t)v| \le \lambda_1(t)|v| \quad \forall v \in H \;\; \forall t \in [0,T], \tag{1.11}$$

with a non-negative continuous *stability function* λ_1. Our main assumption on the nonlinear operator $B(t,\cdot): V \to V'$ is that it satisfies the local Lipschitz condition

$$|A_s^{-1/2}(t)\big(B(t,v) - B(t,\tilde v)\big)| \le \lambda_2(t)|A_s^{1/2}(t)(v-\tilde v)| + \mu_2(t)|v-\tilde v| \quad \forall t \in [0,T], \tag{1.12}$$

for all $v, \tilde v$ in a tube $T_u, T_u := \{v \in V : \min_t \|v - u(t)\| \le 1\}$, around the solution u, defined in terms of the norm $\|\cdot\|$ on V, with a non-negative continuous *stability function* λ_2 and a bounded function μ_2. Clearly, (1.12) holds as an equality with $\mu_2(t) = 0$ for $B(t,\cdot) := \lambda_2(t)A_s(t)$; in particular, the condition $\lambda_2(t) < 1$ ensures local parabolicity of the differential equation in (1.1); actually, depending on the particular scheme we will use for the discretization of (1.1) in time as well as on $\lambda_1(t)$, we will need to assume that $\lambda_2(t)$ is suitably small.

Furthermore, we assume that the operators $A(t), B(t,\cdot): V \to V', t \in [0,T]$, satisfy a Lipschitz condition in t,

$$\|\big(A(t) - A(s)\big)v\|_\star \le L_A|t-s|\,\|v\| \quad \forall s,t \in [0,T] \;\; \forall v \in V, \tag{1.13}$$

and a Lipschitz-like condition, namely

$$\|[B(t,v) - B(t,\tilde v)] - [B(s,v) - B(s,\tilde v)]\|_\star \le L_B|t-s|\,\|v-\tilde v\| \quad \forall s,t \in [0,T], \tag{1.14}$$

for $v, \tilde v \in T_u$, respectively; notice that the Lipschitz-like condition on $B(t,\cdot)$ is local in the second argument. Actually, the Lipschitz conditions (1.13) and (1.14) can easily be relaxed to bounded variation conditions; see Remark 2.1. Notice also that

$$\big[B(t,v) - B(t,\tilde v)\big] - \big[B(s,v) - B(s,\tilde v)\big] = \int_s^t \big[B_t(\tau,v) - B_t(\tau,\tilde v)\big]\mathrm{d}\tau,$$

with $B_t(\tau,\cdot)$ denoting the partial derivative of B with respect to t; thus, it is readily seen that (1.14) is satisfied if $B_t(\tau,\cdot): V \to V'$ satisfies the local Lipschitz condition

$$\|B_t(\tau,v) - B_t(\tau,\tilde v)\|_\star \le L_B\|v-\tilde v\| \quad \forall v,\tilde v \in T_u, \tag{1.15}$$

uniformly in $\tau \in [0,T]$.

Since the implicit scheme (α,β) is A(0)-stable, the product $\alpha_q\beta_q$ is positive. It then follows immediately from the Lax–Milgram lemma that, given a $w \in V'$, the linear equation

$$\alpha_q v + k\beta_q A(t)v = w \tag{1.16}$$

possesses a unique solution $v \in V$, for any fixed $t \in [0,T]$. Therefore, given the starting approximations $U^0,\dots,U^{q-1} \in V$, the approximations $U^q,\dots,U^N \in V$ are well defined by the implicit–explicit scheme (1.3). We refer to [1, §3 and Remark 6.1] for a discussion concerning existence and local uniqueness of approximate solutions $U^q,\dots,U^N\in V$ for the implicit scheme (1.2).

1.2. **The stability result.** We first introduce two constants, $K_{(\alpha,\beta)}$ and $K_{(\alpha,\beta,\gamma)}$, by

$$K_{(\alpha,\beta)} := \sup_{x>0}\max_{\zeta\in\mathscr{K}}\frac{|x\beta(\zeta)|}{|(\alpha+x\beta)(\zeta)|},\quad K_{(\alpha,\beta,\gamma)} := \sup_{x>0}\max_{\zeta\in\mathscr{K}}\frac{|x\gamma(\zeta)|}{|(\alpha+x\beta)(\zeta)|},\tag{1.17}$$

with $\mathscr{K}$ denoting the unit circle in the complex plane, $\mathscr{K}:=\{z\in\mathbb{C}:|z|=1\}$. Under our hypotheses, the constants $K_{(\alpha,\beta)}$ and $K_{(\alpha,\beta,\gamma)}$ are finite; cf. [6, 1]. Actually, with ϑ the largest angle for which the scheme (α,β) is A(ϑ)-stable, we have

$$K_{(\alpha,\beta)} = \frac{1}{\sin\vartheta};\tag{1.18}$$

see [1]. Moreover, for some implicit–explicit multistep schemes, the constants $K_{(\alpha,\beta,\gamma)}$ are explicitly given in [6] and [8].

The main result of this article is as follows:

Theorem 1.1 (Stability of the schemes (1.2) and (1.3))**.** *Let λ_1 and λ_2 be the stability functions of the boundedness condition* (1.11) *and of the local Lipschitz condition* (1.12)*, respectively. Then, under the linear conditions*

$$(\cot\vartheta)\lambda_1(t)+K_{(\alpha,\beta)}\lambda_2(t)<1\quad\forall t\in[0,T]\tag{1.19}$$

and

$$(\cot\vartheta)\lambda_1(t)+K_{(\alpha,\beta,\gamma)}\lambda_2(t)<1\quad\forall t\in[0,T],\tag{1.20}$$

respectively, on the stability functions λ_1 and λ_2, the implicit multistep scheme (1.2) *and the implicit–explicit multistep scheme* (1.3) *are locally stable in the following sense: If $U^0,\dots,U^N\in T_u$ satisfy* (1.2) *and* (1.3)*, respectively, and $V^0,\dots,V^N\in T_u$ satisfy the corresponding perturbed equations*

$$\sum_{i=0}^{q}\big(\alpha_iI+k\beta_iA(t^{n+i})\big)V^{n+i} = k\sum_{i=0}^{q}\beta_iB(t^{n+i},V^{n+i})+kE^n\tag{1.21}$$

and

$$\sum_{i=0}^{q}\big(\alpha_iI+k\beta_iA(t^{n+i})\big)V^{n+i} = k\sum_{i=0}^{q-1}\gamma_iB(t^{n+i},V^{n+i})+kE^n,\tag{1.22}$$

$n=0,\dots,N-q$, respectively, then with $\vartheta^m:=V^m-U^m$, for sufficiently small time step k, we have

$$|\vartheta^n|^2+k\sum_{\ell=0}^{n}\|\vartheta^\ell\|^2\le C\Big\{\sum_{j=0}^{q-1}\big(|\vartheta^j|^2+k\|\vartheta^j\|^2\big)+k\sum_{\ell=0}^{n-q}\|E^\ell\|_\star^2\Big\},\tag{1.23}$$

$n=q,\dots,N$, with a constant C independent of the time step k, the approximations U^n,V^n and the perturbations E^n.

Actually, (1.19) and (1.20) are best possible *linear* sufficient stability conditions on the functions λ_1 and λ_2 in the sense that none of the three coefficients $\cot\vartheta, K_{(\alpha,\beta)}$ and $K_{(\alpha,\beta,\gamma)}$ can be replaced by a smaller coefficient, if we want the schemes (1.2) and (1.3), respectively, to be stable for all initial value problems (1.1) with linear operators $A(t)$ of the form (1.6) satisfying the boundedness condition (1.11) and nonlinear operators $B(t,\cdot)$ satisfying the local Lipschitz condition (1.12); see [3]. Indeed, as we shall see, in the case of the implicit scheme (1.2), a necessary stability condition is linear, namely of the form of (1.19) with strict inequality replaced by nonstrict inequality; therefore, the linear sufficient stability condition (1.19) is sharp, even among possibly nonlinear conditions on the stability functions λ_1 and λ_2.

Stability of implicit–explicit multistep methods under the sufficient stability condition (1.20) was recently established in [3] for a particular subclass of equations of the form (1.1), namely with $A(t) = \big[1 + \mathrm{i}a(t)\big]A_s$, where A_s is a time-independent, positive definite, self-adjoint operator, i is the imaginary unit and a is a real-valued function.

Combining the stability result of Theorem 1.1 with the easily established consistency of the implicit–explicit scheme (1.3), we are led to optimal-order *a priori* error estimates; see Proposition 2.2 in the sequel. These results extend easily to fully discrete schemes if we discretize in space, for instance by the finite element method; cf., e.g., [6, 4]. The error analysis for the implicit schemes (1.2) and the analysis of linearizations of these schemes are left to future work.

We assumed that the stability functions λ_1 and λ_2 are continuous for simplicity; it suffices to assume that the functions on the left-hand sides of the stability conditions (1.19) and (1.20) are uniformly bounded away from 1.

The local stability estimates of Theorem 1.1 are valid in any tube $\widetilde{T}_u \subset V$ of u, defined in terms of other norms, provided the local Lipschitz conditions (1.12) and (1.14) hold in $\widetilde{T}_u$.

An advantage of the implicit schemes is that $K_{(\alpha,\beta)}$ may be much smaller than $K_{(\alpha,\beta,\gamma)}$; see [1, Table 2.1] for the case of the backward difference formula (BDF) methods. A drawback of the implicit schemes, on the other hand, is that to advance in time we need to solve, at each time level, a *nonlinear* equation. Thus, to implement such schemes, we need to linearize, for instance by the Newton method.

1.3. **An example.** Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary $\partial\Omega$, and consider the following initial and boundary value problem, subject to homogeneous Dirichlet boundary conditions,

$$\begin{cases} u_t - \displaystyle\sum_{i,j=1}^{d} \big((a_{ij}(x,t) + \tilde{a}_{ij}(x,t))u_{x_j}\big)_{x_i} = B(t,u) & \text{in} \quad \Omega\times[0,T], \\ u = 0 & \text{on} \quad \partial\Omega\times[0,T], \\ u(\cdot,0) = u^0 & \text{in} \quad \Omega, \end{cases} \tag{1.24}$$

with T positive and $u^0 : \Omega \to \mathbb{C}$ a given initial value. Here, $\mathcal{A}, \widetilde{\mathcal{A}} : \Omega\times[0,T] \to \mathbb{C}^{d,d}$ are uniformly positive definite and Hermitian, and anti-Hermitian matrices, respectively, with smooth entries $a_{ij}(x,t)$ and $\tilde{a}_{ij}(x,t)$, respectively, and $B(t,\cdot)$ are suitable, possibly nonlinear, operators. We assume that (1.24) possesses a smooth solution.

We consider complex-valued functions in Ω and, with standard notation for Sobolev spaces, let $H := L^2 := L^2(\Omega), V := H_0^1 := H_0^1(\Omega)$ and $V' = H^{-1} := H^{-1}(\Omega)$. We then let the time-dependent, linear operators $A_s(t), A_a(t) : V \to V'$ be defined by

$$A_s(t)v := -\sum_{i,j=1}^{d} \big(a_{ij}(\cdot,t)v_{x_j}\big)_{x_i}, \quad A_a(t)v := -\sum_{i,j=1}^{d} \big(\tilde{a}_{ij}(\cdot,t)v_{x_j}\big)_{x_i}, \tag{1.25}$$

and write (1.24) in the form (1.1) with $A(t) := A_s(t) + A_a(t)$ and $B(t,\cdot) : V \to V', t \in [0,T]$, (possibly) nonlinear operators. Notice that the operators $A_s(t)$ and $A_a(t)$ are positive definite and self-adjoint, and anti-self-adjoint, respectively. We shall elaborate on the boundedness condition (1.11) for this example in Section 5.

Some early references for multistep methods for parabolic equations are [31, 14, 15, 32, 18, 19]. The analysis originated in [31], where linear parabolic equations with self-adjoint elliptic operators were considered; spectral techniques were employed. A-stable two-step methods for nonlinear parabolic equations are investigated by the energy technique in [32]. Multistep methods for linear parabolic equations with general time-independent operators are analyzed in [14, 15]; a Cauchy integral representation is used and stability bounds, containing a logarithmic factor, for linear equations with maximally sectorial operators are established. The analysis of [14, 15] is extended via a discrete perturbation argument to linear parabolic equations with time-dependent operators and to quasilinear parabolic equations, respectively, in [18, 19]; spectral theory is employed in the case of self-adjoint operators, whereas the von Neumann inequality is used in the case of A-stable methods; in the case of strongly A(ϑ)-stable methods and operators with numerical range in a sector S_{ϑ_0} of angle $\vartheta_0 < \vartheta$ the analysis relies on the stability bounds of [14, 15].

Implicit–explicit multistep methods, for linear parabolic equations, were introduced and analyzed in [13]; the analysis was extended to nonlinear parabolic equations in [5, 6, 1, 12, 2, 3, 9]. Implicit multistep schemes are studied in [21] for nonlinear stiff differential equations and in [26] for linear parabolic equations with time-dependent operators. The analysis in [21, 26, 6, 1, 2, 3] is based on spectral and Fourier techniques. In contrast, in [13, 5] the energy method is employed; the drawback of the specific analysis is that it does not lead to quantified sufficient stability conditions on the stability functions λ_1 or λ_2. Energy methods for high-order multistep schemes that do lead to quantified sufficient stability conditions were only recently applied to BDF schemes of order up to 5, first in [22] for implicit BDF methods for linear parabolic equations on evolving surfaces, and subsequently in [12] and [2, 9] for implicit and implicit–explicit BDF schemes for quasilinear and nonlinear parabolic equations, respectively.

Multistep and, in particular, BDF methods in Banach spaces are analyzed, e.g., in [24, 16, 17, 11, 10]; the analysis in [24, 16] and [17, 11, 10] relies on semigroup theory and discrete maximal parabolic regularity, respectively; more precisely, in [11] the discrete maximal parabolic regularity is combined with the energy technique.

The efficiency of implicit–explicit BDF methods for nonlinear parabolic equations has been investigated by extensive numerical experiments in, e.g., [7] with very satisfactory results; see also relevant references therein.

For a variety of time-stepping schemes for parabolic equations and their properties, we refer to the classical monograph in this field, namely [28].

An outline of the article is as follows: In Section 2, we first recall the main stability result of [1]; then, combining this result with a suitable decomposition of the operator $A(t)$ in (1.6), we prove Theorem 1.1 and establish optimal-order error estimates. In Section 3, we give necessary stability conditions for the schemes (1.2) and (1.3) and discuss the sharpness of the sufficient stability conditions (1.19) and (1.20); the stability condition for the implicit schemes is sharp even among possibly nonlinear sufficient stability conditions on the stability functions λ_1 and λ_2; whether this is the case also for the implicit–explicit schemes remains open. In Section 4, we present additional sufficient stability conditions. For BDF schemes up to order 5, stability has also been established by the energy technique, under more stringent sufficient stability conditions than the corresponding present ones in the case of the three-, four- and five-step methods. In Section 5, we elaborate on some special cases of parabolic equations of the form (1.24).

2. Proof of the stability result

We first present a stability result of [1] and then prove Theorem 1.1. Our approach combines the main idea of [3] with a discrete perturbation argument and concerns a much wider class of nonlinear parabolic equations. Combining stability and consistency, we establish optimal order error estimates for the implicit–explicit schemes.

2.1. **A known stability result.** In this subsection, we assume that the operators A and B in (1.1) are *time independent*, and the linear operator A is of the form

$$A = A_s + \widetilde{A} \tag{2.1}$$

with $A_s : V \to V'$ a *time-independent*, positive definite, self-adjoint linear operator, and a *general* bounded linear operator $\widetilde{A} : V \to V'$, not necessarily anti-self-adjoint,

$$|A_s^{-1/2}\widetilde{A}v| \le \lambda_1 |A_s^{1/2}v| + \mu_1|v| \quad \forall v \in V, \tag{2.2}$$

with a *stability constant* λ_1 and a constant μ_1.

It is shown in [1] that the implicit scheme (1.2) and the implicit–explicit scheme (1.3), respectively, are locally stable in the tube T_u for the initial value problem (1.1), with operator A as described in this section, provided the stability constants λ_1 and λ_2 in the boundedness condition (2.2) and in the local Lipschitz condition (1.12), with the operator $A_s(t)$ replaced by A_s, are small enough such that

$$K_{(\alpha,\beta)}\lambda_1 + K_{(\alpha,\beta)}\lambda_2 < 1 \tag{2.3}$$

and

$$K_{(\alpha,\beta)}\lambda_1 + K_{(\alpha,\beta,\gamma)}\lambda_2 < 1, \tag{2.4}$$

respectively. Furthermore, (2.3) and (2.4) are best possible *linear* sufficient stability conditions on the constants λ_1 and λ_2 in the sense that none of the coefficients $K_{(\alpha,\beta)}$ and $K_{(\alpha,\beta,\gamma)}$ can be replaced by a smaller coefficient if we want the schemes (1.2) and (1.3), respectively, to be stable for all equations (1.1) satisfying (2.2) and (1.12); see [1, 6]. Notice that both constants $K_{(\alpha,\beta)}$ and $K_{(\alpha,\beta,\gamma)}$ are, for consistent schemes, larger than or equal to 1.

The only difference between the sufficient stability conditions (2.3) and (1.19), as well as between (2.4) and (1.20), is that λ_1 in (2.3) and (2.4), respectively, is replaced by $(\cos\vartheta)\lambda_1$ in (1.19) and (1.20). A geometric interpretation of this difference is that

λ_1 in (2.3) and (2.4) accounts for the operator $\widetilde{A}$ of (2.1), in the 'direction' of which we did not impose any restriction; in contrast, λ_1 in (1.19) and (1.20) accounts for the anti-self-adjoint operator A_a; the fact that A_a is the operator analogue of a purely imaginary number, i.e., a perturbation of the self-adjoint operator A_s in the 'direction' of the imaginary axis, is reflected in (1.19) and (1.20) through the length $(\cos\vartheta)\lambda_1$ of the projection of $\mathrm{i}\lambda_1$ in the direction perpendicular to the boundary of the stability sector S_ϑ; see Figure 2.1.

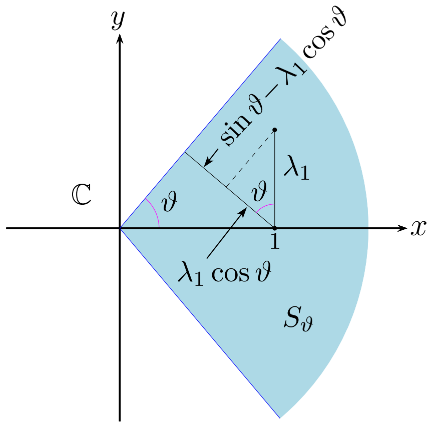

FIGURE 2.1. Geometric interpretation of the stability conditions (1.19) and (1.20): $(\cos\vartheta)\lambda_1$ is the length of the component (projection) of $\mathrm{i}\lambda_1$ in the direction perpendicular to the boundary of the stability sector S_ϑ.

We now present the main stability result of [1] for the implicit–explicit scheme (1.3); see [1, (6.6)] and [6, (2.12)]. The proof of the corresponding stability result for the implicit scheme (1.2) is completely analogous.

Proposition 2.1 (The main stability result of [1])**.** *Assume that the linear operator $A : V \to V'$ is of the form* (2.1)*, and let λ_1 and λ_2 be the stability constants of the boundedness condition* (2.2) *and of the local Lipschitz condition* (1.12) *for the time-independent operator $B : V \to V'$, with the operator $A_s(t)$ replaced by A_s, respectively. Then, under the linear stability condition* (2.4) *on λ_1 and λ_2, the implicit–explicit multistep scheme* (1.3) *is locally stable in the following sense: If $U^0,\dots,U^N,V^0,\dots,V^N \in T_u$ satisfy* (1.3) *and the corresponding perturbed equations* (1.22)*, respectively, then the stability estimate* (1.23) *holds true for the differences $\vartheta^m := V^m - U^m$, with a constant C independent of the time step k, the approximations U^n, V^n and the perturbations E^n.*

2.2. **Proof of Theorem 1.1.** In contrast to Section 2.1, here we consider the initial value problem (1.1) with linear operator $A(t)$ of the form (1.6). In this case, now with λ_1 as in (1.11), the sufficient stability conditions (2.3) and (2.4) can be relaxed to (1.19) and (1.20), respectively, i.e., the first terms on the left-hand sides of (2.3) and (2.4) can be multiplied by $\cos\vartheta$. See the relevant comments in [3].

We will see that Theorem 1.1 for the implicit–explicit scheme (1.3) follows from Proposition 2.1 by using a more favourable decomposition of the operator $A(t)$ and a discrete perturbation argument. The proof of the corresponding stability result for the implicit scheme (1.2) is again completely analogous and is omitted.

The key point in the proof of Theorem 1.1 is the following choice of a decomposition of the operator $A(t)$,

$$A(t) = \widehat{A}_s(t) + \widetilde{A}(t), \quad \text{with} \quad \widehat{A}_s(t) := (1+\eta)A_s(t) \quad \text{and} \quad \widetilde{A}(t) := A_a(t) - \eta A_s(t), \tag{2.5}$$

with η a non-negative quantity that may depend on $\lambda_1(t)$ and $\lambda_2(t)$. We shall see that a suitable choice of η is $\eta := (\tan\vartheta)\lambda_1(t)$, with λ_1 as in (1.11).

Now, in analogy to the anti-self-adjoint operator $\mathcal{A}_a(t) = A_s^{-1/2}(t)A_a(t)A_s^{-1/2}(t)$, see (1.7), with the notation of the decomposition (2.5), we consider the operator

$$\widetilde{\mathcal{A}}(t) := \widehat{A}_s^{-1/2}(t)\widetilde{A}(t)\widehat{A}_s^{-1/2}(t);$$

notice that, in contrast to $\mathcal{A}_a(t)$, the operator $\widetilde{\mathcal{A}}(t)$ is not anti-self-adjoint for positive η, since $\widetilde{A}(t)$ is not anti-self-adjoint. We have the following crucial relation between $\widetilde{\mathcal{A}}(t)$ and $\mathcal{A}_a(t)$,

$$\widetilde{\mathcal{A}}(t) = \widehat{A}_s^{-1/2}(t)\big(A_a(t) - \eta A_s(t)\big)\widehat{A}_s^{-1/2}(t) = \frac{1}{1+\eta}\big(\mathcal{A}_a(t) - \eta I\big),$$

with I the identity operator on H. Now, in view of the anti-self-adjointness of $\mathcal{A}_a(t)$, for any real number $\tilde\eta$ we have

$$|(\mathcal{A}_a(t) - \tilde\eta I)v|^2 = |\mathcal{A}_a(t)v|^2 + \tilde\eta^2|v|^2 \quad \forall v \in H, \tag{2.6}$$

and infer that the norms of the operators $\widetilde{\mathcal{A}}(t)$ and $\mathcal{A}_a(t)$ are related as

$$|\widetilde{\mathcal{A}}(t)|^2 = \frac{1}{(1+\eta)^2}\big(|\mathcal{A}_a(t)|^2 + \eta^2\big). \tag{2.7}$$

From (2.7) and (1.11), we obtain for the norm of $\widetilde{\mathcal{A}}(t)$ the central estimate

$$|\widetilde{\mathcal{A}}(t)v| \le \frac{\sqrt{\lambda_1^2(t) + \eta^2}}{1+\eta}|v| \quad \forall v \in H \ \ \forall t \in [0,T]. \tag{2.8}$$

Furthermore, it is easily seen that the operators $\widehat{A}_s(t)$ and $B(t,\cdot)$ satisfy the estimate

$$|\widehat{A}_s^{-1/2}(t)\big(B(t,v) - B(t,\tilde v)\big)| \le \hat\lambda_2(t)|\widehat{A}_s^{1/2}(t)(v-\tilde v)| + \hat\mu_2(t)|v-\tilde v| \quad \forall v,\tilde v \in T_u, \tag{2.9}$$

with

$$\hat\lambda_2(t) := \frac{\lambda_2(t)}{1+\eta} \quad \text{and} \quad \hat\mu_2(t) := \frac{\mu_2(t)}{\sqrt{1+\eta}}. \tag{2.10}$$

Compare (2.8) with (1.11) and (2.9) with (1.12), respectively.

2.2.1. *Time-independent operators.* We first assume that the operators A and B in (1.1) are time independent. From (2.4) and (2.8), (2.9), (2.10), we then infer that the scheme (1.3) is locally stable for (1.1), with a time-independent operator A of the form (1.6), if λ_1 and λ_2 are such that

$$\frac{1}{\sin\vartheta}\frac{\sqrt{\lambda_1^2 + \eta^2}}{1+\eta} + K_{(\alpha,\beta,\gamma)}\frac{\lambda_2}{1+\eta} < 1, \tag{2.11}$$

for some non-negative η. Now, for $\eta = (\tan\vartheta)\lambda_1$, condition (2.11) reduces to the desired sufficient stability condition (1.20) and the proof is complete. The motivation for this choice of η is that (2.11) can be written in the form

$$\Big[\frac{1}{\sin\vartheta}\sqrt{\lambda_1^2+\eta^2}-\eta\Big] + K_{(\alpha,\beta,\gamma)}\lambda_2 < 1$$

and the expression in brackets attains its minimum at $\eta = (\tan\vartheta)\lambda_1$.

In the case of A-stable implicit methods (α,β), i.e., when $\vartheta = 90°$, it suffices to choose the parameter η in (2.11) large enough.

Let us mention that the sufficient stability condition in [1] is linear; see (2.4). However, since the bound of the norm of $\widetilde{\mathcal{A}}(t)$ in (2.8) is nonlinear in λ_1, we are led to the nonlinear stability condition (2.11).

2.2.2. *Time-dependent operators.* We shall now utilize a discrete perturbation argument to extend our previous stability result to the case of time-dependent operators $A(t), B(t,\cdot) : V \to V', t \in [0,T]$, assuming that they satisfy (1.11) and (1.12) as well as the Lipschitz conditions (1.13) and (1.14), respectively, with respect to t. Although this argument is well known, see, for instance, [18, 19, 26, 23, 11, 10], it is, to the best of our knowledge, employed in the analysis of implicit and implicit–explicit multistep methods for nonlinear parabolic equations by spectral and Fourier techniques for the first time; in particular, it was an essential requirement up to now that the self-adjoint part of the linear operator was time independent.

First, subtracting (1.3) from (1.22), and letting $\vartheta^m := V^m - U^m$, we obtain

$$\sum_{i=0}^{q}\big(\alpha_i I + k\beta_i A(t^{n+i})\big)\vartheta^{n+i} = k\sum_{i=0}^{q-1}\gamma_i\big[B(t^{n+i},V^{n+i}) - B(t^{n+i},U^{n+i})\big] + kE^n, \tag{2.12}$$

$n = 0,\dots,N-q$. For a fixed $m, q \le m \le N$, and for $n = 0,\dots,m-q$, we rewrite (2.12) in the following form

$$\sum_{i=0}^{q}\big(\alpha_i I + k\beta_i A(t^m)\big)\vartheta^{n+i} = k\sum_{i=0}^{q-1}\gamma_i\big[B(t^m,V^{n+i}) - B(t^m,U^{n+i})\big] + k\mathcal{E}^n, \tag{2.13}$$

$n = 0,\dots,m-q$, where

$$\begin{cases} \mathcal{E}^n := E^n + \mathcal{E}_A^n + \mathcal{E}_B^n, \\ \mathcal{E}_A^n := \displaystyle\sum_{i=0}^{q}\beta_i\big[A(t^m) - A(t^{n+i})\big]\vartheta^{n+i}, \\ \mathcal{E}_B^n := -\displaystyle\sum_{i=0}^{q-1}\gamma_i\Big\{\big[B(t^m,V^{n+i}) - B(t^m,U^{n+i})\big] \\ \qquad\qquad - \big[B(t^{n+i},V^{n+i}) - B(t^{n+i},U^{n+i})\big]\Big\}. \end{cases} \tag{2.14}$$

Since the time t is frozen at t^m in both operators $A(t^m)$ and $B(t^m,\cdot)$ in (2.13), and conditions (1.11) and (1.12) are satisfied for $t = t^m$, we can apply the already-established stability estimate for time-independent operators and infer that

$$|\vartheta^m|^2 + k\sum_{\ell=0}^{m}\|\vartheta^\ell\|^2 \le C\Big\{\sum_{j=0}^{q-1}\big(|\vartheta^j|^2 + k\|\vartheta^j\|^2\big) + k\sum_{\ell=0}^{m-q}\|\mathcal{E}^\ell\|_\star^2\Big\}; \tag{2.15}$$

see (1.23) for the case of time-independent operators. Notice also that the constant C in (2.15) is independent of t^m; it only depends on T, the suprema of the function on the left-hand side of the sufficient stability condition (1.20), and the function μ_2 in (1.12).

Our task is now the estimation of the last term on the right-hand side of (2.15) in a suitable way; to this end, we shall use the Lipschitz conditions (1.13) and (1.14) with respect to t. First, we have

$$\|\mathcal{E}_A^\ell\|_\star \le \sum_{i=0}^{q}|\beta_i|\,\|[A(t^m)-A(t^{\ell+i})]\vartheta^{\ell+i}\|_\star,$$

and thus, using the Lipschitz continuity of A in time, see (1.13),

$$\|\mathcal{E}_A^\ell\|_\star \le L_A\sum_{i=0}^{q}|\beta_i|(t^m-t^{\ell+i})\|\vartheta^{\ell+i}\|, \quad \ell=0,\dots,m-q. \tag{2.16}$$

Analogously, using (1.14), we obtain

$$\|\mathcal{E}_B^\ell\|_\star \le L_B\sum_{i=0}^{q-1}|\gamma_i|(t^m-t^{\ell+i})\|\vartheta^{\ell+i}\|, \quad \ell=0,\dots,m-q. \tag{2.17}$$

Let

$$\mathscr{E}^n := \sum_{\ell=0}^{n}\|\mathcal{E}_A^\ell+\mathcal{E}_B^\ell\|_\star^2 \quad\text{and}\quad \varTheta^n := k\sum_{\ell=0}^{n}\|\vartheta^\ell\|^2. \tag{2.18}$$

In view of (2.16) and (2.17), we have

$$\mathscr{E}^{m-q} \le C\sum_{\ell=0}^{m-q}\Big(\sum_{i=0}^{q}(t^m-t^{\ell+i})\|\vartheta^{\ell+i}\|\Big)^2 \le C\sum_{\ell=0}^{m-1}(t^m-t^\ell)^2\|\vartheta^\ell\|^2,$$

whence

$$\mathscr{E}^{m-q} \le Ck\sum_{\ell=0}^{m-1}(m-\ell)^2k\|\vartheta^\ell\|^2,$$

i.e.,

$$\mathscr{E}^{m-q} \le Ck\sum_{\ell=0}^{m-1}(m-\ell)^2(\varTheta^\ell-\varTheta^{\ell-1}), \tag{2.19}$$

with $\varTheta^{-1}:=0$. Now, by summation by parts, we have

$$\sum_{\ell=0}^{m-1}(m-\ell)^2(\varTheta^\ell-\varTheta^{\ell-1}) = \sum_{\ell=0}^{m-1}\big[2(m-\ell)-1\big]\varTheta^\ell,$$

and (2.19) yields

$$\mathscr{E}^{m-q} \le Ck \sum_{\ell=0}^{m-1} (m-\ell)\Theta^\ell \le C \sum_{\ell=0}^{m-1} \Theta^\ell. \tag{2.20}$$

From (2.15) and (2.20), we easily infer that

$$\Theta^m \le C\Big\{ \sum_{j=0}^{q-1} \big(|\vartheta^j|^2 + k\|\vartheta^j\|^2\big) + k \sum_{\ell=0}^{m-q} \|E^\ell\|_\star^2 \Big\} + Ck \sum_{\ell=0}^{m-1} \Theta^\ell. \tag{2.21}$$

Therefore, with a discrete Gronwall inequality, we obtain

$$\Theta^m \le C\Big\{ \sum_{j=0}^{q-1} \big(|\vartheta^j|^2 + k\|\vartheta^j\|^2\big) + k \sum_{\ell=0}^{m-q} \|E^\ell\|_\star^2 \Big\}, \tag{2.22}$$

$m = q, \dots, N$. Combining (2.15) with (2.22), see also (2.20), we get the desired stability estimate (1.23), and the proof of Theorem 1.1 is complete.

Remark 2.1 (Relaxation of the Lipschitz conditions (1.13) and (1.14) to bounded variation conditions)**.** The Lipschitz conditions (1.13) and (1.14), with respect to t, can easily be relaxed to bounded variation conditions,

$$\|\big(A(t) - A(s)\big)v\|_\star \le \big[\sigma_A(t) - \sigma_A(s)\big]\|v\|, \quad 0 \le s \le t \le T, \ \ \forall v \in V, \tag{2.23}$$

and

$$\begin{gathered} \|[B(t,v) - B(t,\tilde v)] - [B(s,v) - B(s,\tilde v)]\|_\star \le \big[\sigma_B(t) - \sigma_B(s)\big]\|v - \tilde v\|, \\ 0 \le s \le t \le T, \ \ \forall v, \tilde v \in T_u, \end{gathered} \tag{2.24}$$

with two increasing functions $\sigma_A, \sigma_B : [0,T] \to \mathbb{R}$. Indeed, it is easily seen that the analogue of (2.19) is in this case

$$\mathscr{E}^{m-q} \le \frac{C}{k} \sum_{\ell=0}^{m-1} \big[\sigma(t^m) - \sigma(t^\ell)\big]^2 (\Theta^\ell - \Theta^{\ell-1}), \tag{2.25}$$

with $\sigma(t) := \sigma_A(t) + \sigma_B(t)$. Then, by summation by parts, we obtain

$$\mathscr{E}^{m-q} \le \frac{C}{k} \sum_{\ell=0}^{m-1} a_\ell \Theta^\ell, \tag{2.26}$$

with $a_\ell := \big[\sigma(t^m) - \sigma(t^\ell)\big]^2 - \big[\sigma(t^m) - \sigma(t^{\ell+1})\big]^2$. From (2.15) and (2.26), we infer that

$$\Theta^m \le C\Big\{ \sum_{j=0}^{q-1} \big(|\vartheta^j|^2 + k\|\vartheta^j\|^2\big) + k \sum_{\ell=0}^{m-q} \|E^\ell\|_\star^2 \Big\} + C \sum_{\ell=0}^{m-1} a_\ell \Theta^\ell. \tag{2.27}$$

Since the sum $\sum_{\ell=0}^{m-1} a_\ell$ is uniformly bounded by a constant independent of m and the time step k,

$$\sum_{\ell=0}^{m-1} a_\ell = \big[\sigma(t^m) - \sigma(0)\big]^2 \le \big[\sigma(T) - \sigma(0)\big]^2,$$

a discrete Gronwall-type argument applied to (2.27) leads again to (2.22), and the stability proof is completed as before.

Remark 2.2 (Slight relaxation of the boundedness condition (1.11))**.** The boundedness condition (1.11) can be slightly relaxed to

$$|\mathcal{A}_a(t)v| \le \lambda_1(t)|v| + \mu_1(t)\|v\|_\star \quad \forall v \in H \ \ \forall t \in [0,T], \tag{2.28}$$

with the non-negative continuous stability function λ_1 as in (1.11) and a bounded function μ_1; μ_1 accounts for lower order terms and does not enter into the sufficient stability conditions (1.19) and (1.20). Compare (2.28) with (2.2). For instance, if A_s is a positive definite self-adjoint operator and $A_a := \mathrm{i}\lambda_1 A_s + \mathrm{i}\mu_1 A_s^{1/2}$, then

$$\mathcal{A}_a := A_s^{-1/2} A_a A_s^{-1/2} = \mathrm{i}\lambda_1 I + \mathrm{i}\mu_1 A_s^{-1/2},$$

whence

$$|\mathcal{A}_a v| \le \lambda_1 |v| + \mu_1 \|v\|_\star \quad \forall v \in H. \tag{2.29}$$

2.3. **Error estimates.** Here, combining the easily established consistency of the implicit–explicit multistep scheme (1.3) with the local stability result of Theorem 1.1, we derive optimal order error estimates. These results extend easily to fully discrete schemes if we discretize in space, for instance by the finite element method; cf., e.g., [6, 4].

For simplicity, we assume that the order of both q-step methods, the implicit (α,β) and the explicit (α,γ), is p, i.e.,

$$\sum_{i=0}^{q} i^\ell \alpha_i = \ell \sum_{i=0}^{q} i^{\ell-1}\beta_i = \ell \sum_{i=0}^{q-1} i^{\ell-1}\gamma_i, \quad \ell = 0,1,\dots,p. \tag{2.30}$$

The consistency error E^n of the scheme (1.3) for the solution u of (1.1), i.e., the amount by which the exact solution misses satisfying (1.3), is given by

$$kE^n = \sum_{i=0}^{q} \big(\alpha_i I + k\beta_i A(t^{n+i})\big)u^{n+i} - k\sum_{i=0}^{q-1}\gamma_i B(t^{n+i}, u^{n+i}), \tag{2.31}$$

$n = 0,\dots,N-q$; we recall that $u^\ell = u(t^\ell)$ are the nodal values of the solution u of (1.1). First, letting

$$E_1^n := \sum_{i=0}^{q} \big[\alpha_i u^{n+i} - k\beta_i u'(t^{n+i})\big], \quad E_2^n := k\sum_{i=0}^{q} (\beta_i - \gamma_i) B(t^{n+i}, u^{n+i}),$$

with $\gamma_q := 0$, and using the differential equation in (1.1), we infer that

$$kE^n = E_1^n + E_2^n. \tag{2.32}$$

Now, by Taylor expanding about t^n, we see that due to the order conditions of the implicit method (α,β), i.e., the first equality in (2.30), and the second equality in (2.30), respectively, the leading terms of order up to p cancel, and we obtain

$$\begin{cases} E_1^n = \dfrac{1}{p!}\displaystyle\sum_{i=1}^{q}\int_{t^n}^{t^{n+i}} (t^{n+i}-s)^{p-1}\big[\alpha_i(t^{n+i}-s) - pk\beta_i\big]u^{(p+1)}(s)\,\mathrm{d}s,\\[2ex] E_2^n = \dfrac{k}{(p-1)!}\displaystyle\sum_{i=1}^{q}(\beta_i-\gamma_i)\int_{t^n}^{t^{n+i}} (t^{n+i}-s)^{p-1}\dfrac{d^p}{dt^p}B(s,u(s))\,\mathrm{d}s. \end{cases} \tag{2.33}$$

Thus, under obvious regularity requirements, we obtain the consistency estimate

$$\max_{0\le n\le N-q} \|E^n\|_\star \le Ck^p. \tag{2.34}$$

Remark 2.3 (A basic difference in the error analysis of implicit and implicit–explicit schemes)**.** For a specific $n \ge q$, the local stability estimate (1.23) is valid for the implicit–explicit scheme (1.3), provided that $U^\ell, V^\ell \in T_u, \ell = 0,\dots,n-1$. Indeed, since γ_q vanishes, no estimate of the difference $B(t^n, U^n) - B(t^n, V^n)$ is needed. This fact plays an important role in the error analysis since it allows us to show inductively that the approximations U^m belong to T_u; see Proposition 2.2. In contrast, in the case of the implicit scheme (1.2), to establish the stability estimate (1.23), for a specific n, we need to assume that $U^\ell, V^\ell \in T_u, \ell = 0,\dots,n$.

Proposition 2.2 (Optimal-order error estimates)**.** *Assume that the stability functions λ_1 and λ_2 of Theorem 1.1 satisfy the stability condition* (1.20)*, that the solution u of* (1.1) *is sufficiently smooth such that the consistency estimate* (2.34) *is valid, and that we are given starting approximations $U^0, U^1,\dots,U^{q-1} \in V$ to $u^0,\dots,u^{q-1}$ such that*

$$\max_{0\le j\le q-1} \big(|u^j - U^j| + k^{1/2}\|u^j - U^j\|\big) \le Ck^p. \tag{2.35}$$

Let $U^q,\dots,U^N \in V$ be recursively defined by the implicit–explicit q-step scheme (1.3)*. Then, there exists a constant C, independent of k, such that for k sufficiently small,*

$$\max_{0\le n\le N} |u(t^n) - U^n| \le Ck^p. \tag{2.36}$$

Proof. Let $e^n := u^n - U^n, n = 0,\dots,N$, and subtract (1.3) from (2.31). According to the consistency estimate (2.34) and our assumption (2.35) on the accuracy of the starting approximations $U^0, U^1,\dots,U^{q-1}$, there exists a constant $C_\star$ such that

$$C\Big\{\sum_{j=0}^{q-1} \big(|e^j|^2 + k\|e^j\|^2\big) + k\sum_{\ell=0}^{N-q} \|E^\ell\|_\star^2\Big\} \le C_\star^2 k^{2p}. \tag{2.37}$$

Next, we shall inductively show that

$$|e^m|^2 + k\sum_{\ell=0}^{m} \|e^\ell\|^2 \le C\Big\{\sum_{j=0}^{q-1} \big(|e^j|^2 + k\|e^j\|^2\big) + k\sum_{\ell=0}^{m-q} \|E^\ell\|_\star^2\Big\}, \tag{2.38}$$

$m = q-1,\dots,N$. This stability estimate completes then the proof since (2.36) is an immediate consequence of (2.38) and (2.37).

Now, the estimate (2.38) is clearly valid for $m = q-1$. Assume inductively that it holds for $m = q-1,\dots,n+q-1, 0 \le n \le N-q$. Then according to (2.37) and the induction hypothesis, we have, for k small enough,

$$\max_{0\le j\le n+q-1} \|e^j\| \le C_\star k^{p-1/2} \le 1, \tag{2.39}$$

and thus $U^j \in T_u, j = 0,\dots,n+q-1$. Therefore, according to our local stability estimate (1.23) with n replaced by $n+q$, see also Remark 2.3, the estimate (2.38) indeed holds for $m = n+q$ as well, and the proof is complete. ☐

3. On the sharpness of the stability conditions (1.19) and (1.20)

In this section, we first prove that the sufficient stability condition (1.19) for the implicit scheme (1.2) is sharp by explicitly constructing suitable parabolic equations. Then, we give necessary stability conditions for the implicit scheme (1.2) as well as for the implicit–explicit scheme (1.3). In the case of the implicit scheme (1.2), the necessary stability condition is linear, and leads us again to the conclusion that the sufficient stability condition (1.19) is sharp. For the implicit–explicit scheme (1.3), the sufficient condition (1.20) is sharp if the implicit method (α, β) is A-stable.

3.1. **Sharpness of the sufficient stability condition (1.19) for the implicit scheme (1.2).** In view of (1.18), the sufficient stability condition (1.19) for the implicit scheme (1.2) can be written as

$$(\cos\vartheta)\lambda_1(t) + \lambda_2(t) < \sin\vartheta \quad \forall t \in [0, T]. \tag{3.1}$$

Notice that the bound $\sin\vartheta$ on the right-hand side is the distance of the number 1 from the boundary of the stability sector S_ϑ of the method. The positive definite self-adjoint operator A_s is the operator analogue of a positive number; thinking of it as normalized, we can view it as the analogue of 1. On the left-hand side of (3.1), the stability function λ_2 is simply added, since it accounts for the perturbation B, in the 'direction' of which we did not impose any restriction. On the other hand, the stability function λ_1 accounts for the anti-self-adjoint operator A_a; the fact that this is a perturbation of the self-adjoint operator A_s in the 'direction' of the imaginary axis, in the sense that anti-self-adjoint operators are the analogues of purely imaginary numbers, is reflected in (3.1) through the coefficient $\cos\vartheta$; see Figure 2.1. The stability condition (3.1) can also be written as $\lambda_2(t) < \sin\vartheta - (\cos\vartheta)\lambda_1(t)$ with the expression on its right-hand side being the distance of the point $1 + \mathrm{i}\lambda_1(t)$ from the boundary of the stability sector S_ϑ of the method; see Figure 2.1.

In the simplest case $\vartheta = 90°$, condition (3.1) reads $\lambda_2(t) < 1$; the last condition is necessary for the parabolicity of $u' + A_s u = \lambda_2(t) A_s u$, with a positive definite self-adjoint operator; in particular, condition (3.1) cannot be relaxed in this case.

We now focus on the interesting case $\vartheta < 90°$ and, assuming that the left-hand side of (1.19) exceeds 1, shall explicitly construct examples of parabolic equations, for which the implicit scheme (1.2) is unstable. Indeed, in that case, we will have $(\cos\hat\vartheta)\lambda_1 + \lambda_2 = \sin\hat\vartheta$ for some $\vartheta < \hat\vartheta < 90°$ (see (3.1)). Consider the ray $\ell_{\hat\vartheta} := \{\rho e^{\mathrm{i}\hat\vartheta}, \rho \ge 0\}$, and let $\hat z_2$ denote the orthogonal projection of $z_1 := 1 + \mathrm{i}\lambda_1$ on $\ell_{\hat\vartheta}$. First, $\lambda_2 = |\hat z_2 - z_1|$; see Figures 2.1 and 4.1. Let now A_s be a positive definite self-adjoint operator with unbounded spectrum and consider the "rotated" operator $\hat z_2 A_s$. The eigenvalues of $\hat z_2 A_s$ lie on the ray $\ell_{\hat\vartheta}$, which is outside the stability sector S_ϑ; therefore, according to the von Neumann criterion, the method (α, β) is unstable for the equation

$$u' + \hat z_2 A_s u = 0. \tag{3.2}$$

More precisely, by definition, for $\hat\vartheta$ sufficiently close to ϑ, the ray $\ell_{\hat\vartheta}$ is not entirely contained in the stability region $\mathcal{S}$ of the method; if λ is an eigenvalue of A_s, then the method is unstable for equation (3.2) for all time steps k such that $k\hat z_2\lambda \notin \mathcal{S}$; since there exists an unbounded sequence of positive eigenvalues of A_s, it is impossible to find a positive k_0 such that the method is stable for equation (3.2) for all time steps $0 < k < k_0$.

Now, we write equation (3.2) in the form

$$u' + A_s u + \mathrm{i}\lambda_1 A_s u = B(u) \tag{3.3}$$

with $B(u) := -(\hat{z}_2 - z_1)A_s u$. It is easily seen that the boundedness condition (1.11) and the Lipschitz condition (1.12) are satisfied for (3.3) with the given constants λ_1 and λ_2. We infer that if the left-hand side in the stability condition (1.19) exceeds the bound on its right-hand side, the method (α, β) is in general unstable.

3.2. **Necessary stability conditions.** Here, we shall give necessary stability conditions for the implicit scheme (1.2) as well as for the implicit–explicit scheme (1.3). The necessary stability conditions are expressed in terms of two suitable functions. In the case of the implicit method (1.2) the corresponding function has a simple form and the necessary stability condition for the implicit scheme (1.2) will lead us to the same conclusion as in the previous subsection, namely that the sufficient stability condition (1.19) is sharp.

3.2.1. *The implicit scheme.* Assume that the implicit method (α, β) is A(ϑ)-stable, with $\vartheta < 90°$ as large as possible. Extending the definition (1.17) of the constant $K_{(\alpha,\beta)}$, let

$$K_{(\alpha,\beta)}(y) := \sup_{x>0} \max_{\zeta\in\mathscr{K}} \frac{|x\beta(\zeta)|}{|\alpha(\zeta) + x(1+\mathrm{i}y)\beta(\zeta)|}, \quad -\tan\vartheta < y < \tan\vartheta. \tag{3.4}$$

Notice that $K_{(\alpha,\beta)}(0) = K_{(\alpha,\beta)}$; see (1.17). Furthermore, since

$$\frac{|x\beta(\bar{\zeta})|}{|\alpha(\bar{\zeta}) + x(1+\mathrm{i}y)\beta(\bar{\zeta})|} = \frac{|x\beta(\zeta)|}{|\alpha(\zeta) + x(1-\mathrm{i}y)\beta(\zeta)|},$$

with $\bar{\zeta}$ denoting the complex conjugate of ζ, $K_{(\alpha,\beta)}$ is an even function of y,

$$K_{(\alpha,\beta)}(y) = K_{(\alpha,\beta)}(-y), \quad |y| < \tan\vartheta.$$

Our task now is to simplify the presentation (3.4) of $K_{(\alpha,\beta)}(y)$. Since the function is even, it suffices to consider the case of non-negative arguments y. Let $d(\zeta) := \alpha(\zeta)/\beta(\zeta)$, for ζ in the unit circle $\mathscr{K}$, represent the points of the *root locus curve* of the method (α, β). The root locus curve is symmetric with respect to the real axis. Since the method is A(ϑ)-stable, the root locus curve lies outside the sector $-S_\vartheta$.

We introduce the parts $\mathscr{K}_y^+$ and $\mathscr{K}_y^-$ of the unit circle $\mathscr{K}$ according to the sign of $\operatorname{Re}\big((1-\mathrm{i}y)d(\zeta)\big)$,

$$\mathscr{K}_y^+ := \{\zeta \in \mathscr{K} : \operatorname{Re}\big((1-\mathrm{i}y)d(\zeta)\big) \geq 0\}, \quad \mathscr{K}_y^- := \{\zeta \in \mathscr{K} : \operatorname{Re}\big((1-\mathrm{i}y)d(\zeta)\big) < 0\},$$

with $d(\zeta)$ the points of the root locus curve; with $0 \leq \varphi < \vartheta$ such that $y = \tan\varphi$, and $d(\zeta) = \rho(\zeta)\mathrm{e}^{\mathrm{i}\psi(\zeta)}, \rho(\zeta) \geq 0, -\pi \leq \psi(\zeta) < \pi$, we have $(1-\mathrm{i}y)d(\zeta) = \frac{\rho(\zeta)}{\cos\vartheta}\mathrm{e}^{\mathrm{i}(\psi(\zeta)-\vartheta)}$, whence $\zeta \in \mathscr{K}_y^+$ and $\zeta \in \mathscr{K}_y^-$, respectively, simply means that $d(\zeta)$ belongs to the rotated by the angle φ non-negative and negative complex half-planes, respectively, in the positive direction. For $\tilde{x} > 0$, we have

$$|1 + \tilde{x}(1-\mathrm{i}y)d(\zeta)|^2 = 1 + 2\tilde{x}\operatorname{Re}\omega(\zeta; y) + \tilde{x}^2|\omega(\zeta; y)|^2 \tag{3.5}$$

with $\omega(\zeta;y) := (1-\mathrm{i}y)d(\zeta)$. Now, for $\zeta \in \mathscr{K}_y^+$, we have $\operatorname{Re}\omega(\zeta;y) \ge 0$, and the infimum of the right-hand side of (3.5), for positive $\tilde{x}$, is 1. For $\zeta \in \mathscr{K}_y^-$, on the other hand, the quadratic polynomial on the right-hand side of (3.5) attains its minimum at

$$\tilde{x}^\star(\zeta;y) := -\frac{\operatorname{Re}\omega(\zeta;y)}{|\omega(\zeta;y)|^2}.$$

Thus, we have

$$\sup_{\tilde{x}>0} \frac{1}{|1+\tilde{x}(1-\mathrm{i}y)d(\zeta)|} = \begin{cases} 1 & \forall \zeta \in \mathscr{K}_y^+, \\ \dfrac{|\omega(\zeta;y)|}{|\operatorname{Im}\omega(\zeta;y)|} = \dfrac{\sqrt{1+y^2}\,|d(\zeta)|}{|\operatorname{Im}\big((1-\mathrm{i}y)d(\zeta)\big)|} & \forall \zeta \in \mathscr{K}_y^-. \end{cases} \tag{3.6}$$

Now, from (3.6) and the definition of $K_{(\alpha,\beta)}(y)$ we infer that

$$K_{(\alpha,\beta)}(y) = \sup_{\zeta\in\mathscr{K}_y^-} \frac{|d(\zeta)|}{|\operatorname{Im}\big((1-\mathrm{i}y)d(\zeta)\big)|}. \tag{3.7}$$

Let $d(\zeta) = -\rho(\zeta)\mathrm{e}^{\mathrm{i}\psi(\zeta)}$, with $\rho(\zeta) \ge 0$ and $-\pi \le \psi(\zeta) < \pi$. Since the method is A($\vartheta$)-stable, we have $|\psi(\zeta)| \ge \vartheta$, and the infimum of $|\psi(\zeta)|$ is ϑ. Now, with $0 \le \varphi < \vartheta$ such that $y = \tan\varphi$, we have

$$(1-\mathrm{i}y)d(\zeta) = -\frac{\rho(\zeta)}{\cos\varphi}\mathrm{e}^{\mathrm{i}(\psi(\zeta)-\varphi)},$$

whence (3.7) takes the form

$$K_{(\alpha,\beta)}(y) = (\cos\varphi) \sup_{\zeta\in\mathscr{K}_y^-} \frac{1}{|\sin\big(\psi(\zeta)-\varphi\big)|}.$$

Since $\cos\big(\psi(\zeta)-\varphi\big) > 0$ for $\zeta \in \mathscr{K}_y^-$, and either $-\pi-\varphi \le \psi(\zeta)-\varphi \le -\vartheta-\varphi$ or $\vartheta-\varphi \le \psi(\zeta)-\varphi \le \pi-\varphi$, it is easily seen that the infimum of $|\sin\big(\psi(\zeta)-\varphi\big)|$ is $\sin(\vartheta-\varphi)$. Thus, the previous relation yields the desired simplified representation

$$K_{(\alpha,\beta)}(y) = \frac{\cos\varphi}{\sin(\vartheta-\varphi)}, \quad y = \tan\varphi. \tag{3.8}$$

Notice that this relation can also be written in the form

$$K_{(\alpha,\beta)}(y) = \frac{1}{\sin\vartheta - \cos\vartheta\tan\varphi} = \frac{1}{\sin\vartheta-(\cos\vartheta)y}, \quad 0 \le y < \tan\vartheta; \tag{3.9}$$

the denominator is the distance of the point $1+\mathrm{i}y$ from the boundary of the stability sector S_ϑ of the method (α,β); see Figure 2.1.

We shall see that

$$K_{(\alpha,\beta)}\big(\lambda_1(t)\big)\lambda_2(t) \le 1 \quad \forall t \in [0,T] \tag{3.10}$$

is a necessary stability condition for the implicit scheme (1.2). As before, (3.10) is necessary if we want (1.2) to be locally stable for all equations satisfying our assumptions with the given stability functions λ_1 and λ_2.

Assume that (3.10) is not satisfied for a certain value $t^\star$ of t; for notational simplicity, we drop the dependence of λ_1 and λ_2 on $t^\star$. Then, for the function k,

$$k(x,\zeta) := \frac{\lambda_2 x\beta(\zeta)}{\alpha(\zeta)+x(1+\mathrm{i}\lambda_1)\beta(\zeta)}, \quad x>0,\ |\zeta| \ge 1, \tag{3.11}$$

we have

$$\exists z \in \mathscr{K},\ x > 0 \quad |k(x,z)| > 1. \tag{3.12}$$

Since

$$\lim_{|\zeta|\to\infty} |k(x,\zeta)| = \frac{\lambda_2 x\beta_q}{|\alpha_q + x(1+\mathrm{i}\lambda_1)\beta_q|} \le \frac{\lambda_2 x\beta_q}{\alpha_q + x\beta_q} < \lambda_2 < 1,$$

we infer that there exists a $\zeta^\star \in \mathbb{C}$ with $|\zeta^\star| > 1$ such that $|k(x,\zeta^\star)| = 1$, i.e.,

$$\frac{\lambda_2 x\beta(\zeta^\star)}{\alpha(\zeta^\star) + x(1+\mathrm{i}\lambda_1)\beta(\zeta^\star)} = \mathrm{e}^{-\mathrm{i}\psi}$$

for a $\psi \in [0, 2\pi)$. Therefore,

$$\alpha(\zeta^\star) + x(1+\mathrm{i}\lambda_1)\beta(\zeta^\star) - \lambda_2 x\mathrm{e}^{\mathrm{i}\psi}\beta(\zeta^\star) = 0. \tag{3.13}$$

Then, with A_s a positive definite self-adjoint operator, choosing the anti-self-adjoint operator $A_a := \mathrm{i}\lambda_1 A_s$ and the linear operator $B := \lambda_2\mathrm{e}^{\mathrm{i}\psi}A_s$, we readily see that the boundedness condition (1.11) and the Lipschitz condition (1.12) are satisfied. According to the von Neumann criterion, a necessary stability condition is that, if ν is an eigenvalue of A_s, the solutions of

$$\sum_{j=0}^{q} \big[\alpha_j + k\nu\big((1+\mathrm{i}\lambda_1)\beta_j - \lambda_2\mathrm{e}^{\mathrm{i}\psi}\beta_j\big)\big]v^{n+j} = 0 \tag{3.14}$$

are bounded; for $k\nu = x$ this is not the case, since in view of (3.13) the root condition is not satisfied. Therefore, the scheme (1.2) is not unconditionally stable.

In view of the presentation (3.9) of $K_{(\alpha,\beta)}(y)$, the necessary stability condition (3.10) for the implicit scheme (1.2) is linear; it takes the form

$$(\cos\vartheta)\lambda_1(t) + \lambda_2(t) \le \sin\vartheta \quad \forall t \in [0,T]. \tag{3.15}$$

Notice that the only difference between the sufficient and necessary stability conditions (1.19) and (3.15), respectively, for the implicit scheme (1.2) is that the former is a strict inequality and the latter a nonstrict inequality.

3.2.2. *The implicit–explicit scheme.* In the case of an A-stable implicit method (α,β), the sufficient stability condition (1.20) for the implicit–explicit scheme (1.3) is sharp. Therefore, as in subsection 3.2.1, we assume that the implicit method (α,β) is A(ϑ)-stable, with $\vartheta < 90°$ as large as possible.

In analogy to the function $K_{(\alpha,\beta)}$, see (3.4), for $-\tan\vartheta < y < \tan\vartheta$, we define the even function $K_{(\alpha,\beta,\gamma)}$ by

$$K_{(\alpha,\beta,\gamma)}(y) := \sup_{x>0}\max_{\zeta\in\mathscr{K}} \frac{|x\gamma(\zeta)|}{|\alpha(\zeta) + x(1+\mathrm{i}y)\beta(\zeta)|}. \tag{3.16}$$

From (3.6) and the definition of $K_{(\alpha,\beta,\gamma)}(y)$, we easily infer that

$$K_{(\alpha,\beta,\gamma)}(y) = \max\Big\{\frac{1}{\sqrt{1+y^2}}\max_{\zeta\in\mathscr{K}_y^+}\frac{|\gamma(\zeta)|}{|\beta(\zeta)|},\ \sup_{\zeta\in\mathscr{K}_y^-}\frac{|d(\zeta)|}{|\operatorname{Im}\big((1-\mathrm{i}y)d(\zeta)\big)|}\frac{|\gamma(\zeta)|}{|\beta(\zeta)|}\Big\}; \tag{3.17}$$

compare to (3.7).

As in the case of the implicit method (1.2), see (3.10),

$$K_{(\alpha,\beta,\gamma)}\big(\lambda_1(t)\big)\lambda_2(t) \le 1 \quad \forall t \in [0,T] \tag{3.18}$$

is a necessary stability condition for the implicit–explicit scheme (1.3). A simple representation of $K_{(\alpha,\beta,\gamma)}$ is unfortunately not available. Also, in contrast to $K_{(\alpha,\beta)}$, the function $K_{(\alpha,\beta,\gamma)}$ is in general not increasing for positive y. Therefore, since

$$K_{(\alpha,\beta,\gamma)}(y)\lambda_2(t) \le 1 \quad \forall t \in [0,T],$$

for all $0 \le y \le \lambda_1(t)$, is clearly also a necessary stability condition, we modify (3.18) as follows: With

$$\widetilde{K}_{(\alpha,\beta,\gamma)}(y) := \sup_{0\le s\le y} K_{(\alpha,\beta,\gamma)}(s), \quad 0 \le y < \tan\vartheta,$$

a necessary stability condition for the implicit–explicit scheme (1.3) is

$$\widetilde{K}_{(\alpha,\beta,\gamma)}\big(\lambda_1(t)\big)\lambda_2(t) \le 1 \quad \forall t \in [0,T]. \tag{3.19}$$

The left-hand sides of (1.20) and (3.19) do not coincide, in general; consequently, in contrast to the implicit schemes (1.2), it remains open, whether the sufficient stability condition (1.20) is best possible also among possibly nonlinear sufficient stability conditions for the implicit–explicit schemes (1.3) when the implicit method (α,β) is not A-stable. The discrepancy between the best possible *linear* sufficient stability condition (1.20) and the corresponding necessary stability condition (3.19) for concrete implicit–explicit schemes (1.3) can be studied at least computationally.

4. Additional sufficient stability conditions

In this section, we discuss additional sufficient stability conditions for schemes (1.2) and (1.3). Here, the non-self-adjointness of $A(t)$ is measured either by the smallest half-angle $\varphi(t)$ of a sector containing its numerical range or by the norm of $\mathcal{A}(t)$ and is estimated by the ratio $\nu(t)/\kappa(t)$.

4.1. Using the norm of $\mathcal{A}(t)$ as a measure of the non-self-adjointness of $A(t)$.

As an alternative to (1.11), let us now use the norm of the operator $\mathcal{A}(t) : H \to H$ as a measure of the non-self-adjointness of the operator $A(t)$,

$$|\mathcal{A}(t)v| \le \tilde{\lambda}_1(t)|v| \quad \forall v \in H \ \forall t \in [0,T]. \tag{4.1}$$

In view of the relation (1.10) between the norms of $\mathcal{A}(t)$ and $\mathcal{A}_a(t)$, it is easily seen that if (1.11) holds, then (4.1) is valid with $\tilde{\lambda}_1(t) = \sqrt{1+[\lambda_1(t)]^2}$, and, conversely, if (4.1) holds, then (1.11) is valid with $\lambda_1(t) = \sqrt{[\tilde{\lambda}_1(t)]^2-1}$. Thus, without loss of generality, we may assume that $\lambda_1(t)$ and $\tilde{\lambda}_1(t)$ are related as follows

$$[\tilde{\lambda}_1(t)]^2 = 1 + [\lambda_1(t)]^2 \quad \forall t \in [0,T]. \tag{4.2}$$

Then, the linear sufficient stability conditions (1.19) and (1.20) for the implicit scheme (1.2) and the implicit–explicit scheme (1.3), respectively, on $\lambda_1(t)$ and $\lambda_2(t)$ transform into the following, nonlinear in $\tilde{\lambda}_1(t)$ and linear in $\lambda_2(t)$, sufficient stability conditions

$$(\cot\vartheta)\sqrt{[\tilde{\lambda}_1(t)]^2-1} + K_{(\alpha,\beta)}\lambda_2(t) < 1 \quad \forall t \in [0,T] \tag{4.3}$$

and

$$(\cot\vartheta)\sqrt{[\tilde{\lambda}_1(t)]^2-1} + K_{(\alpha,\beta,\gamma)}\lambda_2(t) < 1 \quad \forall t \in [0,T], \tag{4.4}$$

respectively. Now,

$$(\cot\vartheta)\sqrt{[\tilde\lambda_1(t)]^2-1}<(\cos\vartheta)\tilde\lambda_1(t) \iff (\cos\vartheta)\tilde\lambda_1(t)<1,$$

and we infer from (4.3) and (4.4), respectively, that

$$(\cos\vartheta)\tilde\lambda_1(t)+K_{(\alpha,\beta)}\lambda_2(t)<1 \quad \forall t\in[0,T] \tag{4.5}$$

and

$$(\cos\vartheta)\tilde\lambda_1(t)+K_{(\alpha,\beta,\gamma)}\lambda_2(t)<1 \quad \forall t\in[0,T] \tag{4.6}$$

are *linear* sufficient stability conditions on $\tilde\lambda_1(t)$ and $\lambda_2(t)$ for the implicit scheme (1.2) and for the implicit–explicit scheme (1.3), respectively. However, as we shall now see, (4.5) and (4.6) are *not* best possible *linear* sufficient stability conditions for the implicit scheme (1.2) and for the implicit–explicit scheme (1.3), respectively, if the method (α,β) is not A-stable. First, in view of (1.8), using the von Neumann stability criterion, it is easily seen that the implicit scheme (α,β) is in general unstable for the linear part of the differential equation in (1.1) (with vanishing nonlinear part B) if $(\cos\vartheta)\tilde\lambda_1$ exceeds 1. Thus, the coefficient $\cos\vartheta$ of $\tilde\lambda_1(t)$ in (4.5) and (4.6) cannot be replaced by a smaller coefficient; but the coefficients $K_{(\alpha,\beta)}$ and $K_{(\alpha,\beta,\gamma)}$ of $\lambda_2(t)$ in (4.5) and (4.6) can be replaced by $(\sin^2\vartheta)K_{(\alpha,\beta)}$ and $(\sin^2\vartheta)K_{(\alpha,\beta,\gamma)}$, respectively; see (4.8) and (4.10) in the sequel; notice that, since the nonlinear sufficient stability condition (4.3) for the implicit scheme (1.2) is sharp, (4.8) is one (among infinitely many) best possible linear sufficient stability condition on $\tilde\lambda_1$ and λ_2 for the implicit scheme (1.2).

First, we rewrite the sharp sufficient stability condition (4.3) in the form

$$K_{(\alpha,\beta)}\lambda_2(t)<1-(\cot\vartheta)\sqrt{[\tilde\lambda_1(t)]^2-1}=:f(\tilde\lambda_1(t)) \quad \forall t\in[0,T] \tag{4.7}$$

and notice that the function f is decreasing and strictly convex in the interval $[1,1/\cos\vartheta]$. Replacing f on the right-hand side of (4.7) by its linear Taylor polynomial, about some point $\tilde x\in(1,1/\cos\vartheta]$, we are led to a *linear* sufficient stability condition for the implicit scheme (1.2), which is best possible in the sense that the corresponding coefficients of $\tilde\lambda_1(t)$ and $\lambda_2(t)$ cannot be replaced by smaller coefficients. For instance, by Taylor expanding about $\tilde x:=1/\cos\vartheta$, we obtain the best possible linear sufficient stability condition

$$K_{(\alpha,\beta)}\lambda_2(t)<\frac{1}{\sin^2\vartheta}\big[1-(\cos\vartheta)\tilde\lambda_1(t)\big],$$

i.e.,

$$(\cos\vartheta)\tilde\lambda_1(t)+(\sin^2\vartheta)K_{(\alpha,\beta)}\lambda_2(t)<1 \quad \forall t\in[0,T]; \tag{4.8}$$

compare to (4.5). This linear stability condition can also be written as

$$(\cos\vartheta)\tilde\lambda_1(t)+(\sin\vartheta)\lambda_2(t)<1 \quad \forall t\in[0,T]. \tag{4.9}$$

The linear sufficient stability condition for the implicit–explicit scheme (1.3) corresponding to (4.8) reads

$$(\cos\vartheta)\tilde\lambda_1(t)+(\sin^2\vartheta)K_{(\alpha,\beta,\gamma)}\lambda_2(t)<1 \quad \forall t\in[0,T]; \tag{4.10}$$

compare to (4.6). Notice that, due to the fact that $\tilde{\lambda}_1(t) \geq 1$ in (4.8) and (4.10), whereas $\lambda_1(t) \geq 0$ in (1.19) and (1.20), the coefficients of $\lambda_2(t)$ in (4.8) and (4.10) are smaller than the corresponding coefficients in (1.19) and (1.20).

4.2. **Using the smallest angle of a sector containing the numerical range of $A(t)$ as a measure of its non-self-adjointness.** As an alternative to the boundedness conditions (1.11) or (4.1), let us now use the smallest half-angle of a sector containing the numerical range of $A(t)$ as a measure of its non-self-adjointness,

$$(A(t)v, v) \in S_{\varphi(t)} \quad \forall v \in V \ \forall t \in [0, T], \tag{4.11}$$

with $0 \leq \varphi(t) < \vartheta$. Then, the boundedness condition (1.11) is satisfied with $\lambda_1(t) = \tan\varphi(t)$; see (1.8). Therefore, the sufficient stability conditions (1.19) and (1.20), for the implicit multistep scheme (1.2) and the implicit–explicit multistep scheme (1.3), respectively, take the form

$$(\cot\vartheta)\tan\varphi(t) + K_{(\alpha,\beta)}\lambda_2(t) < 1 \quad \forall t \in [0, T] \tag{4.12}$$

and

$$(\cot\vartheta)\tan\varphi(t) + K_{(\alpha,\beta,\gamma)}\lambda_2(t) < 1 \quad \forall t \in [0, T], \tag{4.13}$$

respectively. Let us also note that (4.12) can be rewritten as

$$\lambda_2(t) < \frac{1}{\cos\varphi(t)}[\sin\vartheta\cos\varphi(t) - \sin\varphi(t)\cos\vartheta],$$

i.e., as

$$\lambda_2(t) < \frac{\sin(\vartheta - \varphi(t))}{\cos\varphi(t)} \quad \forall t \in [0, T]; \tag{4.14}$$

see Figure 4.1 for the geometric interpretation. Due to the equivalence (1.8), the sufficient stability condition (4.12) for the implicit multistep scheme (1.2) is also sharp.

4.3. **Estimating the non-self-adjointness of $A(t)$ by the ratio $\nu(t)/\kappa(t)$.** The commonly used ratio $\nu(t)/\kappa(t)$ of the boundedness and coercivity functions, see (1.4) and (1.5), is also an estimate of the non-self-adjointness of the operator $A(t)$; notice, however, that this ratio may be a crude estimate of the non-self-adjointness of $A(t)$ since it depends on the choice of the specific norm $\|\cdot\|$ on V; see [2, Example 2.1]. According to (1.4) and (1.5), the time-dependent norms $\|\cdot\|_t$,

$$\|v\|_t := (A_s(t)v, v)^{1/2} = |A_s(t)^{1/2}v| \quad \forall v \in V, \tag{4.15}$$

first used in the stability analysis of BDF methods in [2, 12], are uniformly equivalent to $\|\cdot\|$,

$$\sqrt{\kappa(t)}\,\|v\| \leq \|v\|_t \leq \sqrt{\nu(t)}\,\|v\| \quad \forall v \in V. \tag{4.16}$$

We denote by $\|\cdot\|_{\star,t}$ the corresponding time-dependent dual norm on V',

$$\forall v \in V' \quad \|v\|_{\star,t} := \sup_{\substack{\tilde{v}\in V\\ \tilde{v}\neq 0}} \frac{|(v,\tilde{v})|}{\|\tilde{v}\|_t} = |A_s(t)^{-1/2}v|.$$

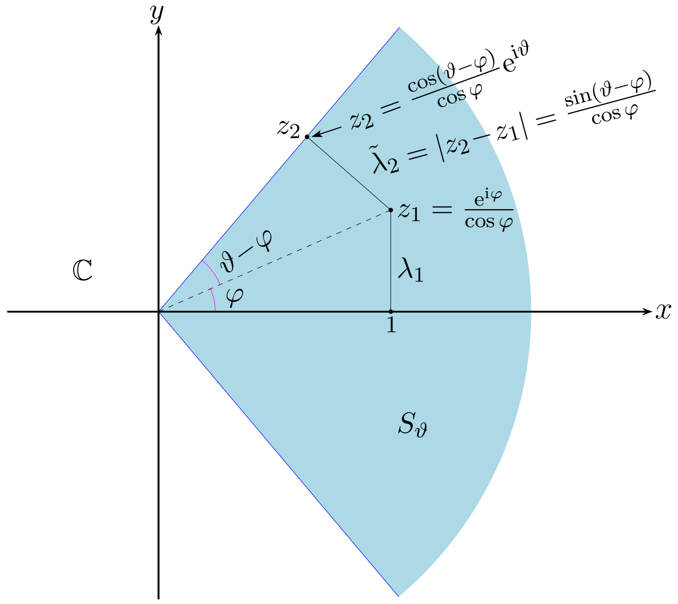

FIGURE 4.1. Geometric interpretation of the stability condition (4.14) for the implicit scheme (α, β). The stability constant λ_2 cannot exceed $\tilde\lambda_2$, the distance of the point $z_1 = 1 + \mathrm{i}\lambda_1$ from the boundary of the stability sector S_ϑ of the method.

Now, for $\hat v \in V, \hat v \neq 0$, using the first inequality in (4.16) and (1.5), with $v := A_s(t)^{-1/2}\hat v$, we have

$$\frac{|\mathcal{A}(t)\hat v|}{|\hat v|} = \frac{\|A(t)v\|_{\star,t}}{\|v\|_t} = \sup_{\substack{\tilde v\in V\\ \tilde v\neq 0}} \frac{|(A(t)v,\tilde v)|}{\|\tilde v\|_t\,\|v\|_t} \le \frac{1}{\kappa(t)} \sup_{\substack{\tilde v\in V\\ \tilde v\neq 0}} \frac{|(A(t)v,\tilde v)|}{\|\tilde v\|\,\|v\|} = \frac{\|A(t)v\|_\star}{\kappa(t)\|v\|} \le \frac{\nu(t)}{\kappa(t)},$$

i.e., $|\mathcal{A}(t)| \le \nu(t)/\kappa(t)$. Consequently, replacing $\tilde\lambda_1(t)$ in (4.3) and (4.4), respectively, by $\nu(t)/\kappa(t)$, we obtain sufficient stability conditions

$$(\cot\vartheta)\frac{1}{\kappa(t)}\sqrt{[\nu(t)]^2 - [\kappa(t)]^2} + K_{(\alpha,\beta)}\lambda_2(t) < 1 \quad \forall t \in [0, T] \tag{4.17}$$

and

$$(\cot\vartheta)\frac{1}{\kappa(t)}\sqrt{[\nu(t)]^2 - [\kappa(t)]^2} + K_{(\alpha,\beta,\gamma)}\lambda_2(t) < 1 \quad \forall t \in [0, T] \tag{4.18}$$

for the implicit scheme (1.2) and the implicit–explicit scheme (1.3), respectively; these conditions are never more advantageous than (4.3) and (4.4), respectively, with $\tilde\lambda_1(t) = |\mathcal{A}(t)|$, and may be much more restrictive than (4.3) and (4.4). Notice, also, that with the norm $\|\cdot\|_t$ on V, (1.4) and (1.5) are satisfied with $\kappa(t) = 1$ and $\nu(t) = |\mathcal{A}(t)|$, that is, the inequality $|\mathcal{A}(t)| \le \nu(t)/\kappa(t)$ holds as an equality for this norm.

Remark 4.1 (Comparison with the energy technique)**.** Stability of implicit as well as of implicit–explicit BDF schemes of order up to 5 can be established also by the energy technique under *linear* sufficient stability conditions on $\tilde\lambda_1(t)$ and $\lambda_2(t)$; see [2, 9, 12]. In the interesting case of the three-, four-, and five-step methods, the stability conditions of [2, 9, 12] are more stringent than the corresponding conditions (4.5) and (4.6). In particular, the conditions of [2, 9, 12] are, for high order BDF schemes, not best possible linear sufficient stability conditions.

5. Application to example (1.24)

In this section we briefly discuss the boundedness condition (1.11) in the case of the initial and boundary value problem (1.24). We will see that the bound $\lambda_1(t)$ is the spectral radius of a matrix; in two simple cases more explicit forms of $\lambda_1(t)$ can be derived. To save space, we do not discuss the Lipschitz condition (1.12) here, since it is a reformulation of corresponding conditions used in, e.g., [6, 12, 7, 3]; we refer to these articles and references therein for examples of operators satisfying (1.12) or modifications thereof in tubes defined in terms of L^∞-based Sobolev norms.

Let $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ denote the L^2 inner product and the product L^2 inner product in $\boldsymbol{L}^2 := (L^2)^d := \big(L^2(\Omega)\big)^d$, respectively; we denote by $|\cdot|$ both corresponding norms. With the notation of subsection 1.3, let us introduce in $V, V = H_0^1 = \mathscr{D}(A^{1/2}(t)), 0 \le t \le T$, the time-dependent norms $\|\cdot\|_t$ by $\|v\|_t := |A_s^{1/2}(t)v|$. We identify H with its dual, and denote by V' the dual of $V, V' = H^{-1}$, and by $\|\cdot\|_{\star,t}$ the time-dependent dual norms on $V', \|v\|_{\star,t} := |A_s^{-1/2}(t)v|$. We use the notation $(\cdot,\cdot)$ also for the antiduality pairing between V' and V; then $\|v\|_t = (A_s(t)v,v)^{1/2}$ and $\|v\|_{\star,t} = (v, A_s^{-1}(t)v)^{1/2}$.

We shall see that the boundedness condition (1.11) is satisfied with

$$\lambda_1(t) := \max_{x\in\overline{\Omega}} \rho\big(S(x,t)\big), \quad t\in[0,T], \tag{5.1}$$

where $\rho(\cdot)$ is the spectral radius of the antihermitian matrices $S(x,t) := \mathcal{A}^{-1/2}(x,t)$ $\widetilde{\mathcal{A}}(x,t)\mathcal{A}^{-1/2}(x,t)$.

First, clearly,

$$(A_s v,\tilde v) = \sum_{i,j=1}^d (a_{ij}v_{x_j},\tilde v_{x_i}) = \sum_{i=1}^d \Big(\sum_{j=1}^d a_{ij}v_{x_j},\tilde v_{x_i}\Big) = \sum_{i=1}^d \big((\mathcal{A}\nabla v)_i,\tilde v_{x_i}\big),$$

i.e.,

$$(A_s(t)v,\tilde v) = \langle \mathcal{A}(\cdot,t)\nabla v,\nabla\tilde v\rangle.$$

In particular, $(A_s(t)v,v) = \langle \mathcal{A}(\cdot,t)\nabla v,\nabla v\rangle$, whence

$$\|v\|_t = |A_s^{1/2}(t)v| = |\mathcal{A}^{1/2}(\cdot,t)\,\nabla v|. \tag{5.2}$$

Now,

$$(A_a(t)v,\tilde v) = \langle \widetilde{\mathcal{A}}(\cdot,t)\nabla v,\nabla\tilde v\rangle = \langle S(\cdot,t)\mathcal{A}^{1/2}(\cdot,t)\,\nabla v,\mathcal{A}^{1/2}(\cdot,t)\,\nabla\tilde v\rangle,$$

whence

$$|(A_a(t)v,\tilde v)| \le \max_{x\in\overline{\Omega}} \|S(x,t)\|_2\,|\mathcal{A}^{1/2}(\cdot,t)\,\nabla v|\,|\mathcal{A}^{1/2}(\cdot,t)\,\nabla\tilde v|, \tag{5.3}$$

with $\|\cdot\|_2$ denoting the spectral (Euclidean) norm of a matrix. Therefore,

$$|(A_a(t)v,\tilde v)| \le \max_{x\in\overline{\Omega}} \|S(x,t)\|_2\,|A_s^{1/2}(t)v|\,|A_s^{1/2}(t)\tilde v|, \quad t\in[0,T],$$

and thus, since the spectral norms of the antihermitian matrices $S(x,t)$ are equal to their spectral radii,

$$|A_s^{-1/2}(t)A_a(t)v| \le \lambda_1(t)|A_s^{1/2}(t)v| \quad \forall v \in V \ \forall t \in [0,T],$$

i.e.,

$$|A_s^{-1/2}(t)A_a(t)A_s^{-1/2}(t)v| \le \lambda_1(t)|v| \quad \forall v \in H \ \forall t \in [0,T], \tag{5.4}$$

with λ_1 as given in (5.1). Thus (1.11) is satisfied with the function λ_1 of (5.1).

Next we consider two special cases. In the first case, (5.1) takes a very simple form; in the second, we give an explicit form of the matrices $S(x,t)$; their spectral radii can then easily be computed.

5.1. **Two special cases.** *First case*: Assume first that

$$\widetilde{\mathcal{A}}(x,t) = \mathrm{i}a(x,t)\mathcal{A}(x,t), \quad x \in \Omega, \quad 0 \le t \le T, \tag{5.5}$$

with a a smooth real-valued function, $a : \overline{\Omega} \times [0,T] \to \mathbb{R}$. Then, $S(x,t) = \mathrm{i}a(x,t)I_d$ with I_d the $d \times d$ unit matrix; thus,

$$\lambda_1(t) = \max_{x\in\overline{\Omega}} |a(x,t)| \quad \forall t \in [0,T]. \tag{5.6}$$

The analysis of implicit–explicit multistep methods in this particular case, with a matrix $\mathcal{A}$ independent of t and a function a independent of x, was the subject of [3]. If

$$\mathcal{A}(x,t) = a(x,t)I_d, \ \ \widetilde{\mathcal{A}}(x,t) = \mathrm{i}\tilde{a}(x,t)I_d, \quad x \in \Omega, \quad 0 \le t \le T, \tag{5.7}$$

with a and $\tilde{a}$ smooth real-valued functions, $a, \tilde{a} : \overline{\Omega} \times [0,T] \to \mathbb{R}$, then the parabolic equation in (1.24) takes the form

$$u_t - \nabla \cdot \big((a(x,t) + \mathrm{i}\tilde{a}(x,t))\nabla u\big) = B(t,u).$$

In this case

$$\widetilde{\mathcal{A}}(x,t) = \mathrm{i}\frac{\tilde{a}(x,t)}{a(x,t)}\mathcal{A}(x,t), \quad x \in \Omega, \quad 0 \le t \le T,$$

and, according to (5.6),

$$\lambda_1(t) = \max_{x\in\overline{\Omega}} \frac{|\tilde{a}(x,t)|}{a(x,t)} \quad \forall t \in [0,T]. \tag{5.8}$$

Second case: Here we shall consider the general case in two space dimensions, $d = 2$. It is well known that

$$\mathcal{A}^{1/2} = \frac{1}{\sqrt{\operatorname{tr}\mathcal{A} + 2\sqrt{\det\mathcal{A}}}}\big(\mathcal{A} + \sqrt{\det\mathcal{A}}\, I_2\big), \tag{5.9}$$

with $\operatorname{tr}\mathcal{A} := a_{11} + a_{22}$ the trace of $\mathcal{A}$. An easy way to check this is by means of the Cayley–Hamilton theorem, which in the case of 2×2 matrices yields $\mathcal{A}^2 + (\det\mathcal{A})I_2 = (\operatorname{tr}\mathcal{A})\mathcal{A}$; then,

$$\big(\mathcal{A} + \sqrt{\det\mathcal{A}}\, I_2\big)^2 = \big(\operatorname{tr}\mathcal{A} + 2\sqrt{\det\mathcal{A}}\,\big)\mathcal{A},$$

and (5.9) follows. Furthermore, multiplying (5.9) by $\mathcal{A}^{-1/2}$, we easily see that

$$\mathcal{A}^{-1/2} = \frac{1}{\sqrt{\det\mathcal{A}}\,\sqrt{\operatorname{tr}\mathcal{A} + 2\sqrt{\det\mathcal{A}}}}\Big(\big(\operatorname{tr}\mathcal{A} + \sqrt{\det\mathcal{A}}\,\big)I_2 - \mathcal{A}\Big). \tag{5.10}$$

Therefore, the antihermitian matrices $S(x,t) = \mathcal{A}^{-1/2}(x,t)\widetilde{\mathcal{A}}(x,t)\mathcal{A}^{-1/2}(x,t)$ take in this case the form

$$S = \frac{1}{\det \mathcal{A}\,\big(\operatorname{tr}\mathcal{A} + 2\sqrt{\det \mathcal{A}}\,\big)}\big(c_{\mathcal{A}}^2\widetilde{\mathcal{A}} - c_{\mathcal{A}}(\mathcal{A}\widetilde{\mathcal{A}} + \widetilde{\mathcal{A}}\mathcal{A}) + \mathcal{A}\widetilde{\mathcal{A}}\mathcal{A}\big), \tag{5.11}$$

with the constant $c_{\mathcal{A}} := \operatorname{tr}\mathcal{A} + \sqrt{\det \mathcal{A}}$.

References

1. G. Akrivis, *Implicit–explicit multistep methods for nonlinear parabolic equations*, Math. Comp. **82** (2013) 45–68.
2. G. Akrivis, *Stability of implicit–explicit backward difference formulas for nonlinear parabolic equations*, SIAM J. Numer. Anal. **53** (2015) 464–484.
3. G. Akrivis, *Stability properties of implicit–explicit multistep methods for a class of nonlinear parabolic equations*, Math. Comp. **85** (2016) 2217–2229.
4. G. Akrivis and M. Crouzeix, *Linearly implicit methods for nonlinear parabolic equations*, Math. Comp. **73** (2004) 613–635.
5. G. Akrivis, M. Crouzeix, and Ch. Makridakis, *Implicit–explicit multistep finite element methods for nonlinear parabolic problems*, Math. Comp. **67** (1998) 457–477.
6. G. Akrivis, M. Crouzeix, and Ch. Makridakis, *Implicit–explicit multistep methods for quasilinear parabolic equations*, Numer. Math. **82** (1999) 521–541.
7. G. Akrivis, A. Kalogirou, D. T. Papageorgiou, and Y.-S. Smyrlis, *Linearly implicit schemes for multi-dimensional Kuramoto–Sivashinsky type equations arising in falling film flows*, IMA J. Numer. Anal. **36** (2016) 317–336.
8. G. Akrivis and F. Karakatsani, *Modified implicit–explicit BDF methods for nonlinear parabolic equations*, BIT Numer. Math. **43** (2003) 467–483.
9. G. Akrivis and E. Katsoprinakis, *Backward difference formulae: new multipliers and stability properties for parabolic equations*, Math. Comp. **85** (2016) 2195–2216.
10. G. Akrivis and B. Li, *Maximum norm analysis of implicit–explicit backward difference formulae for nonlinear parabolic equations*, IMA J. Numer. Anal. (2017) doi:10.1093/imanum/drx008
11. G. Akrivis, B. Li, and C. Lubich, *Combining maximal regularity and energy estimates for time discretizations of quasilinear parabolic equations*, Math. Comp. **86** (2017) 1527–1552.
12. G. Akrivis and C. Lubich, *Fully implicit, linearly implicit and implicit–explicit backward difference formulae for quasi-linear parabolic equations*, Numer. Math. **131** (2015) 713–735.
13. M. Crouzeix, *Une méthode multipas implicite–explicite pour l'approximation des équations d'évolution paraboliques*, Numer. Math. **35** (1980) 257–276.
14. M. Crouzeix and P.-A. Raviart, *Approximation des équations d'évolution linéaires par des méthodes à pas multiples*, C. R. Acad. Sci. Paris Sér. A-B **28** (1976) A367–A370.
15. M. Crouzeix and P.-A. Raviart, *Approximation d'équations d'évolution linéaires par des méthodes multipas.* Étude numérique des grands systèmes (Proc. Sympos., Novosibirsk, 1976) 133–150. Méthodes Math. Inform. **7**, Dunod, Paris, 1978.
16. C. González and C. Palencia, *Stability of time-stepping methods for abstract time-dependent parabolic problems*, SIAM J. Numer. Anal. **35** (1998) 973–989.
17. B. Kovács, B. Li, and C. Lubich, *A-stable time discretizations preserve maximal parabolic regularity*, SIAM J. Numer. Anal. **54** (2016) 3600–3624.
18. M.-N. Le Roux, *Semi-discrétisation en temps pour les équations d'évolution paraboliques lorsque l'opérateur dépend du temps*, RAIRO Anal. Numér. **13** (1979) 119–137.
19. M.-N. Le Roux, *Méthodes multipas pour des équations paraboliques non linéaires*, Numer. Math. **35** (1980) 143–162.
20. J.–L. Lions, E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications.* Vol. I. Springer–Verlag, New York Heidelberg, 1972.
21. C. Lubich, *On the convergence of multistep methods for nonlinear stiff differential equations*, Numer. Math. **58** (1991) 839–853. Erratum: Numer. Math. **61** (1992) 277–279.

22. C. Lubich, D. Mansour, and C. Venkataraman, *Backward difference time discretization of parabolic differential equations on evolving surfaces*, IMA J. Numer. Anal. **33** (2013) 1365–1385.
23. C. Lubich and A. Ostermann, *Runge–Kutta approximations of quasi-linear parabolic equations*, Math. Comp. **64** (1995) 601–627.
24. C. Palencia, *Stability of rational multistep approximations of holomorphic semigroups*, Math. Comp. **64** (1995) 591–599.
25. S. Salsa, *Partial Differential Equations in Action. From Modelling to Theory.* Springer–Verlag Italia, Milano, 2008.
26. G. Savaré, *$A(\Theta)$-stable approximations of abstract Cauchy problems*, Numer. Math. **65** (1993) 319–335.
27. B. Simon, *Operator Theory. A Comprehensive Course in Analysis, Part 4.* AMS, Providence, R. I., 2015.
28. V. Thomée, *Galerkin Finite Element Methods for Parabolic Problems.* 2nd ed., Springer–Verlag, Berlin, 2006.
29. J. Wloka, *Partial Differential Equations.* Cambridge University Press, Cambridge, 1987.
30. A. Yagi, *Abstract Parabolic Evolution Equations and Their Applications.* Springer–Verlag, Berlin Heidelberg, 2010.
31. M. Zlámal, *Finite element multistep discretizations of parabolic boundary value problems*, Math. Comp. **29** (1975) 350–359.
32. M. Zlámal, *Finite element methods for nonlinear parabolic equations*, RAIRO Anal. Numér. **11** (1977) 93–107.

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