Colinear Coloring and Colinear Graphs*

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Abstract

Motivated by the notion of linear coloring on simplicial complexes, recently introduced in the context of algebraic topology, and the framework through which it was studied, we introduce the colinear coloring on graphs. We provide an upper bound for the chromatic number $\chi(G)$, for any graph $G$, and show that $G$ can be colinearly colored in polynomial time by proposing a simple algorithm. The colinear coloring of a graph $G$ is a vertex coloring such that two vertices can be assigned the same color, if their corresponding clique sets are associated by the set inclusion relation (a clique set of a vertex $u$ is the set of all maximal cliques containing $u$); the colinear chromatic number $\lambda(G)$ of $G$ is the least integer $k$ for which $G$ admits a colinear coloring with $k$ colors. Based on the colinear coloring, we define the $\chi$-colinear and $\alpha$-colinear properties and characterize known graph classes in terms of these properties. Consequently, we study the graphs which are characterized completely by the $\chi$-colinear or $\alpha$-colinear property, and conclude that these graphs form two new classes of perfect graphs, which we call colinear and linear graphs. We provide characterizations for colinear and linear graphs and prove structural properties.

Keywords: Colinear coloring, colinear coloring algorithm, colinear graphs, linear graphs, co-chordal graphs, chordal graphs, threshold graphs, quasi-threshold graphs.

1 Introduction

Framework-Motivation. A colinear coloring of a graph $G$ is a coloring of its vertices such that two vertices are assigned different colors, if their corresponding clique sets are not associated by the set inclusion relation; a clique set of a vertex $u$ is the set of all maximal cliques in $G$ containing $u$. The colinear chromatic number $\lambda(G)$ of $G$ is the least integer $k$ for which $G$ admits a colinear coloring with $k$ colors.

Motivated by the definition of linear coloring on simplicial complexes associated to graphs, first introduced by Civan and Yalçın [6] in the context of algebraic topology, we studied linear colorings on simplicial complexes which can be represented by a graph. In particular, we studied the linear coloring problem on a simplicial complex, namely independence complex $I(G)$ of a graph $G$. The independence complex $I(G)$ of a graph $G$ can always be represented by a graph and, more specifically, is identical to the complement graph $\overline{G}$ of the graph $G$; indeed, the facets of $I(G)$ are exactly the maximal cliques of $\overline{G}$. The outcome of this study was the definition of the colinear coloring of a graph $G$; the colinear coloring of a graph $G$ is a coloring of $G$ such that for any set of vertices taking the same color, the collection of their clique sets can be linearly ordered by inclusion. Note that, the two definitions cannot always be considered as identical since not in all cases a simplicial complex can be represented by a graph; such an example is the neighborhood

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complex $\mathcal{N}(G)$ of a graph $G$. Recently, Civan and Yalçın [6] studied the linear coloring of the neighborhood complex $\mathcal{N}(G)$ of a graph $G$ and proved that the linear chromatic number of $\mathcal{N}(G)$ gives an upper bound for the chromatic number $\chi(G)$ of the graph $G$. This approach lies in a general framework met in algebraic topology.

In the context of algebraic topology, one can find much work done on providing boundaries for the chromatic number of an arbitrary graph $G$, by examining the topology of the graph through different simplicial complexes associated to the graph. This domain was motivated by Kneser’s conjecture, which was posed in 1955, claiming that “if we split the $n$-subsets of a $(2n + k)$-element set into $k + 1$ classes, one of the classes will contain two disjoint $n$-subsets” [12]. Kneser’s conjecture was first proved by Lovász in 1978, with a proof based on graph theory, by rephrasing the conjecture into “the chromatic number of Kneser’s graph $KG_{n,k}$ is $k + 2$” [13]. Many more topological and combinatorial proofs followed, the interest of which extends beyond the original conjecture [16]. Although Kneser’s conjecture is concerned with the chromatic numbers of certain graphs (Kneser graphs), the proof methods that are known provide lower bounds for the chromatic number of any graph [14]. Thus, this initiated the application of topological tools in studying graph theory problems and more particularly in graph coloring problems [7].

The interest to provide boundaries for the chromatic number $\chi(G)$ of an arbitrary graph $G$ through the study of different simplicial complexes associated to $G$, which is found in algebraic topology bibliography, drove the motivation for defining the colinear coloring on the graph $G$ and studying the relation between the chromatic number $\chi(G)$ and the colinear chromatic number $\lambda(\overline{G})$. We show that for any graph $G$, $\lambda(\overline{G})$ is an upper bound for $\chi(G)$. The interest of this result lies on the fact that we present a colinear coloring algorithm that can be applied to any graph $G$ and provides an upper bound $\lambda(G)$ for the chromatic number of the graph $G$, i.e., $\chi(G) \leq \lambda(G)$; in particular, it provides a proper vertex coloring of $G$ using $\lambda(G)$ colors. Additionally, recall that a known lower bound for the chromatic number of any graph $G$ is the clique number $\omega(G)$ of $G$, i.e., $\chi(G) \geq \omega(G)$. Motivated by the definition of perfect graphs, for which $\chi(G_A) = \omega(G_A)$ holds $\forall A \subseteq V(G)$, it was interesting to study those graphs for which the equality $\chi(G) = \lambda(G)$ holds, and even more those graphs for which this equality holds for every induced subgraph.

Our Results. In this paper, we first introduce the colinear coloring of a graph $G$ and study the relation between the colinear coloring of $G$ and the proper vertex coloring of $G$. We prove that, for any graph $G$, a colinear coloring of $G$ is a proper vertex coloring of $G$ and, thus, $\lambda(G)$ is an upper bound for $\chi(G)$, i.e., $\chi(G) \leq \lambda(G)$. We present a colinear coloring algorithm that can be applied to any graph $G$. Motivated by these results and the Perfect Graph Theorem [9], we study those graphs for which the equality $\chi(G) = \lambda(G)$ holds for every induced subgraph and characterize known graph classes in terms of the $\chi$-colinear and the $\alpha$-colinear properties. A graph $G$ has the $\chi$-colinear property if its chromatic number $\chi(G)$ equals to the colinear chromatic number $\lambda(G)$ of its complement graph $\overline{G}$, and the equality holds for every induced subgraph of $G$, i.e., $\chi(G_A) = \lambda(G_A)$, $\forall A \subseteq V(G)$; a graph $G$ has the $\alpha$-colinear property if its stability number $\alpha(G)$ equals to its colinear chromatic number $\lambda(G)$, and the equality holds for every induced subgraph of $G$, i.e., $\alpha(G_A) = \lambda(G_A)$, $\forall A \subseteq V(G)$. Note that the stability number $\alpha(G)$ of a graph $G$ is the greatest integer $r$ for which $G$ contains an independent set of size $r$. We show that the class of threshold graphs is characterized by the $\chi$-colinear property and the class of quasi-threshold graphs is characterized by the $\alpha$-colinear property.

Moreover, it was interesting to study those graphs which are characterized completely by the $\chi$-colinear or the $\alpha$-colinear property. The outcome of this study was to conclude that these graphs form two new classes of perfect graphs, which we call colinear and linear graphs, respectively. We also provide characterizations for colinear and linear graphs and prove structural properties. More specifically, we show that the class of colinear graphs is a subclass of co-chordal graphs, a superclass of threshold graphs, and is distinguished from the class of split graphs. Additionally, we infer that
linear graphs form a subclass of chordal graphs and a superclass of quasi-threshold graphs. We also prove that any $P_6$-free chordal graph, which is not a linear graph, properly contains a $k$-sun as an induced subgraph. However, the $k$-sun is not a forbidden induced subgraph for the class of linear graphs and, thus, linear graphs form a superclass of the class of $P_6$-free strongly chordal graphs.

Road Map. The rest of this paper is organized as follows. In Section 2 we provide some basic definitions and we review the terminology and the notation that we use throughout the paper. In Section 3 we define the colinear coloring on graphs, while in Section 4 we present a polynomial time algorithm for colinear coloring which can be applied to any graph $G$ and provides an upper bound for the chromatic number $\chi(G)$ of the graph $G$. In Section 5 we define the $\chi$-colinear and $\alpha$-colinear properties and characterize known graph classes in terms of these properties. Based on these results, in Section 6 we study the graphs which are characterized completely by the $\chi$-colinear or $\alpha$-colinear property and, thus, define two new classes of perfect graphs, which we call colinear and linear graphs. Characterizations and structural properties of linear graphs are proved in Section 7. Some concluding remarks follow.

2 Preliminaries

Let $G$ be a finite undirected graph with no loops or multiple edges. We denote by $V(G)$ and $E(G)$ the vertex set and edge set of $G$. The subgraph of a graph $G$ induced by a subset $S$ of vertices of $G$ is denoted by $G[S]$.

An edge is a pair of distinct vertices $x, y \in V(G)$, and is denoted by $xy$ if $G$ is an undirected graph and by $\overrightarrow{xy}$ if $G$ is a directed graph. For a set $A \subseteq V(G)$ of vertices of the graph $G$, the subgraph of $G$ induced by $A$ is denoted by $G[A]$. Additionally, the cardinality of a set $A$ is denoted by $|A|$. For a given vertex ordering $(v_1, v_2, \ldots, v_n)$ of a graph $G$, the subgraph of $G$ induced by the set of vertices $\{v_i, v_{i+1}, \ldots, v_n\}$ is denoted by $G_i$. The set $N(v) = \{u \in V(G) : uv \in E(G)\}$ is called the open neighborhood of the vertex $v \in V(G)$ in $G$, sometimes denoted by $N_G(v)$ for clarity reasons. The set $N[v] = N(v) \cup \{v\}$ is called the closed neighborhood of the vertex $v \in V(G)$ in $G$.

In a graph $G$, the length of a path is the number of edges in the path. The distance $d(v, u)$ from vertex $v$ to vertex $u$ is the minimum length of a path from $v$ to $u$; $d(v, u) = \infty$ if there is no path from $v$ to $u$.

The greatest integer $r$ for which a graph $G$ contains an independent set of size $r$ is called the independence number or otherwise the stability number of $G$ and is denoted by $\alpha(G)$. The cardinality of the vertex set of the maximum clique in $G$ is called the clique number of $G$ and is denoted by $\omega(G)$. A proper vertex coloring of a graph $G$ is a coloring of its vertices such that no two adjacent vertices are assigned the same color. The chromatic number $\chi(G)$ of $G$ is the least integer $k$ for which $G$ admits a proper vertex coloring with $k$ colors. For the numbers $\omega(G)$ and $\chi(G)$ of an arbitrary graph $G$ the inequality $\omega(G) \leq \chi(G)$ holds. In particularly, $G$ is a perfect graph if the equality $\omega(G_A) = \chi(G_A)$ holds $\forall A \subseteq V(G)$.

Next, definitions of some graph classes mentioned throughout the paper follow. A graph is called a chordal graph if it does not contain an induced subgraph isomorphic to a chordless cycle of four or more vertices. A graph is called a co-chordal graph if it is the complement of a chordal graph [9]. A hole is a chordless cycle $C_n$ if $n \geq 5$; the complement of a hole is an antihole. A graph $G$ is a split graph if there is a partition of the vertex set $V(G) = K + I$, where $K$ induces a clique in $G$ and $I$ induces an independent set. Split graphs are characterized as $(2K_2, C_4, C_5)$-free. Threshold graphs are defined as those graphs where stable subsets of their vertex sets can be distinguished by using a single linear inequality. Threshold graphs were introduced by Chvátal and Hammer [5] and characterized as $(2K_2, P_4, C_4)$-free. Quasi-threshold graphs are characterized
as the \( (P_4, C_4) \)-free graphs and are also known in the literature as trivially perfect graphs [9, 15]. A graph is strongly chordal if it admits a strong elimination ordering. Strongly chordal graphs were introduced by Farber in [8] and are characterized completely as those chordal graphs which contain no \( k \)-sun as an induced subgraph. For more details on basic definitions in graph theory refer to [2, 9].

3 Colinear Coloring on Graphs

In this section we define the colinear coloring of a graph \( G \), and we prove some properties of such a coloring. It is worth noting that these properties have been also proved for the linear coloring of the neighborhood complex \( \mathcal{N}(G) \) in [6].

Definition 3.1 Let \( G \) be a graph and let \( v \in V(G) \). The clique set of a vertex \( v \) is the set of all maximal cliques of \( G \) containing \( v \) and is denoted by \( \mathcal{C}_G(v) \).

Definition 3.2 Let \( G \) be a graph and let \( k \) be an integer. A surjective map \( \kappa : V(G) \to \{1, 2, \ldots, k\} \) is called a \( k \)-colinear coloring of \( G \) if the collection \( \{\mathcal{C}_G(v) : \kappa(v) = i\} \) is linearly ordered by inclusion for all \( i \in \{1, 2, \ldots, k\} \), where \( \mathcal{C}_G(v) \) is the clique set of \( v \), or, equivalently, for two vertices \( v, u \in V(G) \), if \( \kappa(v) = \kappa(u) \) then either \( \mathcal{C}_G(v) \subseteq \mathcal{C}_G(u) \) or \( \mathcal{C}_G(v) \supseteq \mathcal{C}_G(u) \). The least integer \( k \) for which \( G \) is \( k \)-colinear colorable is called the colinear chromatic number of \( G \) and is denoted by \( \lambda(G) \).

Next, we study the colinear coloring on graphs and its association to the proper vertex coloring. In particular, we show that for any graph \( G \) the colinear chromatic number of \( \overline{G} \) is an upper bound for \( \chi(G) \).

Proposition 3.1 Let \( G \) be a graph. If \( \kappa : V(G) \to \{1, 2, \ldots, k\} \) is a \( k \)-colinear coloring of \( \overline{G} \), then \( \kappa \) is a coloring of the graph \( G \).

Proof. Let \( G \) be a graph and let \( \kappa : V(G) \to \{1, 2, \ldots, k\} \) be a \( k \)-colinear coloring of \( \overline{G} \). From Definition 3.2, we have that for any two vertices \( v, u \in V(G) \), if \( \kappa(v) = \kappa(u) \) then either \( \mathcal{C}_G(v) \subseteq \mathcal{C}_G(u) \) or \( \mathcal{C}_G(v) \supseteq \mathcal{C}_G(u) \) holds. Without loss of generality, assume that \( \mathcal{C}_G(v) \subseteq \mathcal{C}_G(u) \) holds. Consider a maximal clique \( C \in \mathcal{C}_G(v) \). Since \( \mathcal{C}_G(v) \subseteq \mathcal{C}_G(u) \), we have \( C \in \mathcal{C}_G(u) \). Thus, both \( u, v \in C \) and therefore \( uv \in E(G) \) and \( uv \notin E(G) \). Hence, any two vertices assigned the same color in a \( k \)-colinear coloring of \( G \) are not neighbors in \( G \). Concluding, any \( k \)-colinear coloring of \( \overline{G} \) is a coloring of \( G \).

It is therefore straightforward to conclude the following.

Corollary 3.1 For any graph \( G \), \( \lambda(\overline{G}) \geq \chi(G) \).

In Figure 1 we depict a colinear coloring of the well known graphs \( 2K_2, C_4 \) and \( P_4 \), using the least possible colors, and show the relation between the chromatic number \( \chi(G) \) of each graph \( G \in \{2K_2, C_4, P_4\} \) and the colinear chromatic number \( \lambda(\overline{G}) \).

Proposition 3.2 Let \( G \) be a graph. A coloring \( \kappa : V(G) \to \{1, 2, \ldots, k\} \) of \( G \) is a \( k \)-colinear coloring of \( G \) if and only if either \( N_G[u] \subseteq N_G[v] \) or \( N_G[u] \supseteq N_G[v] \) holds in \( G \), for every \( u, v \in V(G) \) with \( \kappa(u) = \kappa(v) \).

Proof. Let \( G \) be a graph and let \( \kappa : V(G) \to \{1, 2, \ldots, k\} \) be a \( k \)-colinear coloring of \( G \). We will show that either \( N_G[u] \subseteq N_G[v] \) or \( N_G[u] \supseteq N_G[v] \) holds in \( G \) for every \( u, v \in V(G) \) with \( \kappa(u) = \kappa(v) \). Consider two vertices \( v, u \in V(G) \), such that \( \kappa(u) = \kappa(v) \). Since \(\kappa\) is a colinear
As the DAG associated to the graph \( x \leq y \) is constructed on a partially ordered set of elements \((V(D), \leq)\), such that for some \( x, y \in V(D)\), \( x \leq y \Rightarrow N_G[x] \subseteq N_G[y] \). Throughout the paper we refer to the constructed directed acyclic graph as the DAG associated to the graph \( G \) and denote it by \( D_G \).

Thus, a vertex coloring of \( G \) is unique up to isomorphism.

Additionally, it is easy to see that \( D \) is a transitive directed acyclic graph. Indeed, by definition \( D \) is constructed on a partially ordered set of elements \((V(D), \leq)\), such that for some \( x, y \in V(D)\), \( x \leq y \Rightarrow N_G[x] \subseteq N_G[y] \). Throughout the paper we refer to the constructed directed acyclic graph as the DAG associated to the graph \( G \) and denote it by \( D_G \).
The algorithm for colinear coloring. The proposed algorithm computes a colinear coloring and the colinear chromatic number of a graph $G$. The algorithm works as follows:

(i) **compute** the closed neighborhood set of every vertex of $G$ and, then, find the inclusion relations among the neighborhood sets and construct the DAG $D_G$ associated to the graph $G$.

(ii) **find** a minimum path cover $\mathcal{P}(D_G)$, and its size $\rho(D_G)$, of the transitive DAG $D_G$ (e.g. see [1,11]).

(iii) **assign** a color $\kappa(v)$ to each vertex $v \in V(D_G)$, such that vertices belonging to the same path of $\mathcal{P}(D_G)$ are assigned the same color and vertices of different paths are assigned different colors; this is a surjective map $\kappa : V(D_G) \rightarrow \rho(D_G))$.

(iv) **return** the value $\kappa(v)$ for each vertex $v \in V(D_G)$ and the size $\rho(D_G)$ of the minimum path cover of $D_G$; $\kappa$ is a colinear coloring of $G$ and $\rho(D_G)$ equals the colinear chromatic number $\lambda(G)$ of $G$.

**Correctness of the algorithm.** Let $G$ be a graph and let $D_G$ be the DAG associated to the graph $G$, which is unique up to isomorphism. Consider the value $\kappa(v)$ for each vertex $v \in V(D_G)$ returned by the algorithm and the size $\rho(D_G)$ of a minimum path cover of $D_G$. We show that the surjective map $\kappa : V(D_G) \rightarrow \rho(D_G))$ is a colinear coloring of the vertices of $G$, and prove that the size $\rho(D_G)$ of a minimum path cover $\mathcal{P}(D_G)$ of the DAG $D_G$ is equal to the colinear chromatic number $\lambda(G)$ of the graph $G$.

**Proposition 4.1** Let $G$ be a graph and let $D_G$ be the DAG associated to the graph $G$. A colinear coloring of the graph $G$ can be obtained by assigning a particular color to all vertices of each path of a path cover of the DAG $D_G$. Moreover, the size $\rho(D_G)$ of a minimum path cover $\mathcal{P}(D_G)$ of the DAG $D_G$ equals to the colinear chromatic number $\lambda(G)$ of the graph $G$.

**Proof.** Let $G$ be a graph, $D_G$ be the DAG associated to $G$, and let $\mathcal{P}(D_G)$ be a minimum path cover of $D_G$. The size $\rho(D_G)$ of the DAG $D_G$, equals to the minimum number of directed paths in $D_G$ needed to cover the vertices of $D_G$ and, thus, the vertices of $G$. Now, consider a coloring $\kappa : V(D_G) \rightarrow \{1,2,\ldots,k\}$ of the vertices of $D_G$, such that vertices belonging to the same path are assigned the same color and vertices of different paths are assigned different colors. Therefore, we have $\rho(D_G)$ colors and $\rho(D_G)$ sets of vertices, one for each color. For every set of vertices belonging to the same path, their corresponding closed neighborhood sets can be linearly ordered by inclusion. Indeed, consider a path in $D_G$ with vertices $\{v_1, v_2, \ldots, v_m\}$ and edges $v_i v_{i+1}$ for $i \in \{1,2,\ldots,m\}$. From the construction of $D_G$, it holds that $\forall i,j \in \{1,2,\ldots,m\}$, $v_i v_j \in E(D_G) \Rightarrow N_G[v_i] \subseteq N_G[v_j]$. In other words, the corresponding neighborhood sets of the vertices belonging to a path in $D_G$ are linearly ordered by inclusion. Thus, the coloring $\kappa$ of the vertices of $D_G$ gives a colinear coloring of $G$. This colinear coloring $\kappa$ is optimal, uses $k = \rho(D_G)$ colors, and gives the colinear chromatic number $\lambda(G)$ of the graph $G$. Indeed, suppose that there exists a different colinear coloring $\kappa' : V(D_G) \rightarrow \{k'\}$ of $G$ using $k'$ colors, such that $k' < k$. For every color given in $\kappa'$, consider a set consisted of the vertices assigned that color. It is true that for the vertices belonging to the same set, their neighborhood sets are linearly ordered by inclusion. Therefore, these vertices can belong to the same path in $D_G$. Thus, each set of vertices in $G$ corresponds to a path in $D_G$ and, additionally, all vertices of $G$ (and therefore of $D_G$) are covered. This is a path cover of $D_G$ of size $\rho(D_G) = k' < k = \rho(D_G)$, which is a contradiction since $\mathcal{P}(D_G)$ is a minimum path cover of $D_G$. Therefore, we conclude that the colinear coloring $\kappa : V(D_G) \rightarrow \rho(D_G))$ is optimal and, hence, $\rho(D_G) = \lambda(G)$.

6
5 Graphs having the $\chi$-colinear and $\alpha$-colinear Properties

In Section 3 we showed that for any graph $G$, the colinear chromatic number $\chi(G)$ of the graph $\overline{G}$ is an upper bound for the chromatic number $\chi(G)$ of $G$, i.e., $\chi(G) \leq \chi(\overline{G})$. Motivated by the Perfect Graph Theorem [9], in this section we exploit our results on colinear coloring and we study those graphs for which the equality $\chi(G) = \chi(\overline{G})$ holds for every induced subgraph. The outcome of this study was the definition of the following two graph properties and the characterization of known graph classes in terms of these properties.

- **$\chi$-colinear property.** A graph $G$ has the $\chi$-colinear property if for every induced subgraph $G_A$ of the graph $G$, $\chi(G_A) = \chi(\overline{G_A})$, $A \subseteq V(G)$.

- **$\alpha$-colinear property.** A graph $G$ has the $\alpha$-colinear property if for every induced subgraph $G_A$ of a graph $G$, $\alpha(G_A) = \chi(G_A)$, $A \subseteq V(G)$.

Next, we show that the class of threshold graphs is characterized by the $\chi$-colinear property and the class of quasi-threshold graphs is characterized by the $\alpha$-colinear property. We also show that any graph that has the $\chi$-colinear property is perfect; actually, we show that any graph that has the $\chi$-colinear property is a co-chordal graph. We first give some definitions and show some interesting results.

**Definition 5.1** An edge $uv$ of a graph $G$ is called actual if neither $N_G[u] \subseteq N_G[v]$ nor $N_G[u] \supseteq N_G[v]$. The set of all actual edges of $G$ will be denoted by $E_a(G)$.

**Definition 5.2** A graph $G$ is called quasi-threshold if it has no induced subgraph isomorphic to a $C_4$ or a $P_4$ or, equivalently, if it contains no actual edges.

More details on actual edges and characterizations of quasi-threshold graphs through a classification of their edges can be found in [15]. The following result directly follows from Definition 5.1 and Proposition 3.2.

**Proposition 5.1** Let $\kappa : V(G) \to \{1, 2, \ldots, k\}$ be a $k$-colinear coloring of the graph $G$. If the edge $uv \in E(G)$ is an actual edge of $G$, then $\kappa(u) \neq \kappa(v)$.

Based on Definition 5.1, the $\chi$-colinear property, and Proposition 5.1, we prove the following result.

**Proposition 5.2** Let $G$ be a graph and let $F$ be the graph such that $V(F) = V(G)$ and $E(F) = E(G) \cup E_a(\overline{G})$. The graph $G$ has the $\chi$-colinear property if $\chi(G_A) = \omega(F_A)$, $\forall A \subseteq V(G)$. 


Proof. Let $G$ be a graph and let $F$ be a graph such that $V(F) = V(G)$ and $E(F) = E(G) \cup E_\alpha(\overline{G})$, where $E_\alpha(\overline{G})$ is the set of all actual edges of $\overline{G}$. By definition, $G$ has the $\chi$-colinear property if $\chi(G_A) = \chi(\overline{G}_A)$, $\forall A \subseteq V(G)$. It suffices to show that $\lambda(\overline{G}_A) = \omega(F_A)$, $\forall A \subseteq V(G)$. From Definition 3.2, it is easy to see that two vertices which are not connected by an edge in $\overline{G}_A$ belong necessarily to different cliques and, thus, they cannot receive the same color in a colinear coloring of $\overline{G}_A$. In other words, the vertices which are connected by an edge in $G_A$ cannot take the same color in a colinear coloring of $\overline{G}_A$. Moreover, from Proposition 5.1 vertices which are endpoints of actual edges in $\overline{G}_A$ cannot take the same color in a colinear coloring of $\overline{G}_A$.

Next, we construct the graph $F_A$ with vertex set $V(F_A) = V(G_A)$ and edge set $E(F_A) = E(G_A) \cup E_\alpha(\overline{G}_A)$, where $E_\alpha(\overline{G}_A)$ is the set of all actual edges of $\overline{G}_A$. Every two vertices in $F_A$, which have to take a different color in a colinear coloring of $\overline{G}_A$ are connected by an edge. Thus, the size of the maximum clique in $F_A$ equals to the size of the maximum set of vertices which pairwise must take a different color in $\overline{G}_A$, i.e., $\omega(F_A) = \lambda(\overline{G}_A)$ holds for all $A \subseteq V(G)$. Concluding, $G$ has the $\chi$-colinear property if $\chi(G_A) = \omega(F_A)$, $\forall A \subseteq V(G)$. 

Taking into consideration Proposition 5.2 and the structure of the edge set $E(F) = E(G) \cup E_\alpha(\overline{G})$ of the graph $F$, it is easy to see that $E(F) = E(G)$ if $G$ has no actual edges. Actually, this will be true for all induced subgraphs, since if $G$ is a quasi-threshold graph then $G_A$ is also a quasi-threshold graph for all $A \subseteq V(G)$. Thus, $\chi(G_A) = \omega(F_A)$, $\forall A \subseteq V(G)$. Therefore, the following result holds.

Corollary 5.1 Let $G$ be a graph. If $\overline{G}$ is quasi-threshold, then $G$ has the $\chi$-colinear property.

Using Corollary 5.1 we can prove a more interesting result.

Proposition 5.3 Any threshold graph has the $\chi$-colinear property.

Proof. Let $G$ be a threshold graph. It has been proved that an undirected graph $G$ is a threshold graph if and only if $G$ and its complement $\overline{G}$ are quasi-threshold graphs [15]. From Corollary 5.1, if $\overline{G}$ is quasi-threshold then $G$ has the $\chi$-colinear property. Concluding, if $G$ is threshold, then $\overline{G}$ is quasi-threshold and thus $G$ has the $\chi$-colinear property. 

We note that the proof that any threshold graph $G$ has the $\chi$-colinear property can be also obtained by showing that any coloring of a threshold graph $G$ is a colinear coloring of $\overline{G}$ by using Proposition 3.2, Corollary 3.1, the fact that $N_G(u) = V(G) \setminus N_\overline{G}[u]$, and the property that $N(u) \subseteq N[v]$ or $N(v) \subseteq N[u]$ for any two vertices $u, v$ of $G$. However, Proposition 5.2 and Corollary 5.1 actually give us a stronger result, since the class of quasi-threshold graphs is a superclass of the class of threshold graphs.

The following result is even more interesting, since it shows that any graph that has the $\chi$-colinear property is a perfect graph.

Proposition 5.4 Any graph that has the $\chi$-colinear property is a co-chordal graph.

Proof. Let $G$ be a graph that has the $\chi$-colinear property. It has been shown that a co-chordal graph is $(2K_2$, antihole)-free [9]. To show that any graph $G$ that has the $\chi$-colinear property is a co-chordal graph we will show that if $G$ has a $2K_2$ or an antihole as induced subgraph, then $G$ does not have the $\chi$-colinear property. Since by definition a graph $G$ has the $\chi$-colinear property if the equality $\chi(G_A) = \chi(\overline{G}_A)$ holds for every induced subgraph $G_A$ of $G$, it suffices to show that the graphs $2K_2$ and antihole do not have the $\chi$-colinear property.

The graph $2K_2$ does not have the $\chi$-colinear property, since $\chi(2K_2) = 2 \neq 4 = \lambda(C_4)$; see Figure 1. Now, consider the graph $G = \overline{C_n}$ which is an antihole of size $n \geq 5$. We will show that $\chi(G) \neq \lambda(\overline{G})$. It follows that $\lambda(\overline{G}) = \lambda(C_n) = n \geq 5$, i.e., if the graph $\overline{G} = C_n$ is to
be colored colinearly, every vertex has to take a different color. Indeed, assume that a colinear coloring $\kappa : V(G) \to \{1, 2, \ldots, k\}$ of $G = C_n$ exists such that for some $u_i, u_j \in V(G)$, $i \neq j$, $1 \leq i, j \leq n$, $\kappa(u_i) = \kappa(u_j)$. Since $u_i, u_j$ are vertices of a hole, their neighborhoods in $G$ are $N[u_i] = \{u_{i-1}, u_i, u_{i+1}\}$ and $N[u_j] = \{u_{j-1}, u_j, u_{j+1}\}$, $2 \leq i, j \leq n - 1$. For $i = 1$ or $i = n$, $N[u_1] = \{u_n, u_2\}$ and $N[u_n] = \{u_{n-1}, u_1\}$. Since $\kappa(u_i) = \kappa(u_j)$, from Proposition 3.2 we obtain that one of the inclusion relations $N[u_i] \subseteq N[u_j]$ or $N[u_i] \supseteq N[u_j]$ must hold in $G$. Obviously this is possible if and only if $i = j$, for $n \geq 5$; this is a contradiction to the assumption that $i \neq j$. Thus, no two vertices in a hole take the same color in a colinear coloring. Therefore, $\lambda(G) = n$. It suffices to show that $\chi(G) < n$. It is easy to see that for the antihole $C_n$, $\deg(u) = n - 3$, for every vertex $u \in V(G)$. Brook’s theorem [3] states that for an arbitrary graph $G$ and for all $u \in V(G)$, $\chi(G) \leq \max\{d(u) + 1\} = (n - 3) + 1 = n - 2$. Therefore, $\chi(G) \leq n - 2 < n = \lambda(G)$. Thus the antihole $C_n$ does not have the $\chi$-colinear property.

We have showed that the graphs $2K_2$ and antihole do not have the $\chi$-colinear property. It follows that any graph that has the $\chi$-colinear property is $(2K_2, \text{antihole})$-free and, thus, any graph that has the $\chi$-colinear property is a co-chordal graph. \[\Box\]

Since graphs having the $\chi$-colinear property are perfect, it follows that any graph $G$ having the $\chi$-colinear property satisfies $\chi(G_A) = \omega(G_A) = \alpha(G_A)$, $\forall A \subseteq V(G)$. Therefore, the following result holds.

**Proposition 5.5** A graph $G$ has the $\alpha$-colinear property if and only if the graph $\overline{G}$ has the $\chi$-colinear property.

From Corollary 5.1 and Proposition 5.5 we can obtain the following result.

**Proposition 5.6** Any quasi-threshold graph has the $\alpha$-colinear property.

In this section we defined the $\chi$-colinear and $\alpha$-colinear properties and characterized known graph classes in terms of these properties. Based on these results, we next study the graphs which are characterized completely by the $\chi$-colinear or $\alpha$-colinear property.

## 6 Colinear and Linear Graphs

In Section 5 we showed that any threshold graph has the $\chi$-colinear property and any quasi-threshold graph has the $\alpha$-colinear property. In this section we study the graphs that are characterized completely by the $\chi$-colinear property or the $\alpha$-colinear property. We call these graphs colinear and linear graphs and as we next show they constitute two new classes of perfect graphs.

**Definition 6.1** A graph $G$ is called colinear if and only if $G$ has the $\chi$-colinear property, i.e., $\chi(G_A) = \lambda(G_A)$, $\forall A \subseteq V(G)$. A graph $G$ is called linear if and only if $G$ has the $\alpha$-colinear property, i.e., $\alpha(G_A) = \lambda(G_A)$, $\forall A \subseteq V(G)$.
From Proposition 5.3 we know that any threshold graph is a colinear graph. However, not any colinear graph is a threshold graph. Indeed, Chvátal and Hammer [5] showed that threshold graphs are $(2K_2, P_4, C_4)$-free and, thus, the graphs $P_4$ and $C_4$ are colinear graphs but they are not threshold graphs (see Figure 1). Therefore, we directly obtain the following result concerning the class of colinear graphs.

**Proposition 6.1** Colinear graphs form a superclass of threshold graphs.

Moreover, from Proposition 5.4 we have that any colinear graph is a co-chordal graph. However, the reverse is not always true. For example, the graph $G$ in Figure 2 is a co-chordal graph but it is not a colinear graph. Indeed, $\chi(G) = 4$ and $\lambda(G) = 5$. It is easy to see that this graph is also a split graph. Moreover, not any colinear graph is a split graph, since the graph $C_4$ is colinear but it is not a split graph. However, there exist split graphs which are also colinear graphs; an example is the graph $C_3$. Recall that a graph $G$ is a split graph if there is a partition of the vertex set $V(G) = K + I$, where $K$ induces a clique in $G$ and $I$ induces an independent set; split graphs are characterized as $(2K_2, C_4, C_5)$-free graphs. Thus, the following result holds.

**Proposition 6.2** Colinear graphs form a subclass of co-chordal graphs.

We have proved that colinear graphs do not contain a $2K_2$ or an antihole. Note that, since $C_5 = C_5$ and also the chordless cycle $C_n$ is not $2K_2$-free for $n \geq 6$, it is easy to see that colinear graphs are hole-free. In addition, the graph $P_6$ is not a colinear graph (see Figure 3). Thus, we obtain the following result.

**Proposition 6.3** If a graph $G$ is colinear, then $G$ is a $(2K_2, \text{antihole}, P_6)$-free graph.

From Proposition 5.6 we obtain that any quasi-threshold graph is a linear graph. Again, the reverse is not always true; an example is the graph $P_4$, which is a linear graph but not a quasi-threshold graph. Therefore, the following result holds.

**Proposition 6.4** Linear graphs form a superclass of quasi-threshold graphs.

From Propositions 6.3 and 5.5 we obtain that linear graphs are $(C_4, \text{hole}, P_6)$-free graphs. Therefore, it follows that any linear graph is chordal. However, the reverse is not always true, i.e., not any chordal graph graph is linear; an obvious example is the graph $P_6$. Another interesting example is the complement $\overline{G}$ of the graph illustrated in Figure 2, which is a chordal graph but not a linear graph. Indeed, $\alpha(\overline{G}) = 4$ and $\lambda(\overline{G}) = 5$. It is easy to see that this graph is also a split graph. Moreover, not any linear graph is a split graph, since the graph $2K_2$ is linear but it is not a split graph. However, there exist split graphs that are linear graphs; an example is the graph $C_4$. Therefore, the following result holds.

**Proposition 6.5** Linear graphs form a subclass of chordal graphs.
Figure 4: Illustrating the inclusion relations among the classes of colinear graphs, linear graphs, and other classes of perfect graphs.

Proposition 6.5 implies that linear graphs are perfect graphs and, thus, it follows that any linear graph satisfies $\alpha(G_A) = \omega(G_A) = \chi(G_A), \forall A \subseteq V(G)$. Therefore, from Corollary 3.1 we obtain the following characterization.

**Proposition 6.6** Linear graphs are those graphs $G$ for which the colinear chromatic number achieves its theoretical lower bound in every induced subgraph of $G$.

From the results proved in this section, we conclude that colinear and linear graphs form two new classes of perfect graphs. The inclusion relations among the classes of colinear graphs, linear graphs, and other subclasses of co-chordal and chordal graphs are depicted in Figure 4.

In the next section we prove structural properties of linear graphs by studying the relation between the class of strongly chordal graphs, which is a known subclass of chordal graphs [2, 8], and the class of linear graphs.

## 7 Structural Properties

In this section we prove structural properties of linear graphs, by investigating the structure of their forbidden induced subgraphs. In particular, we prove that any $P_6$-free chordal graph which is not a linear graph properly contains a $k$-sun as an induced subgraph. Let us give the definitions of a $k$-sun and an incomplete $k$-sun. An incomplete $k$-sun $S_k$ ($k \geq 3$) is a chordal graph on $2k$ vertices whose vertex set can be partitioned into two sets, $U = \{u_1, u_2, \ldots, u_k\}$ and $W = \{w_1, w_2, \ldots, w_k\}$, so that $W$ is an independent set, and $w_i$ is adjacent to $u_j$ if and only if $i = j$ or $i = j + 1 \pmod{k}$. A $k$-sun is an incomplete $k$-sun $S_k$ in which $U$ is a complete graph.

The following definitions and results on strongly chordal graphs given in [4, 8], turn up to be useful in proving structural properties of linear graphs.

**Definition 7.1 (Farber [8])** A vertex ordering $(v_1, v_2, \ldots, v_n)$ is a strong elimination ordering of a graph $G$ iff $\sigma$ is a perfect elimination ordering and also has the property that for each $i$, $j$, $k$ and $\ell$, if $i < j$, $k < \ell$, $v_k, v_\ell \in N[v_i]$, and $v_k \in N[v_j]$, then $v_\ell \in N[v_j]$. A graph is strongly chordal iff it admits a strong elimination ordering.

A vertex $v$ of a graph $G$ is called simple if $\{N[x] : x \in N[v]\}$ is linearly ordered by inclusion. It has been proved that a strong elimination ordering of a graph $G$ is a vertex ordering $(v_1, v_2, \ldots, v_n)$
such that for every $i \in \{1, 2, \ldots, n\}$ the vertex $v_i$ is simple in $G_i$ and also $N_{G_i}[v_i] \subseteq N_{G_i}[v_k]$ whenever $i \leq \ell \leq k$ and $v_\ell, v_k \in N_{G_i}[v_i]$ [4]; recall that for a given vertex ordering $(v_1, v_2, \ldots, v_n)$ of a graph $G$, we denote by $G_i$ the subgraph of $G$ induced by the set of vertices $\{v_1, v_{i+1}, \ldots, v_n\}$. Additionally, a graph $G$ is strongly chordal if and only if every induced subgraph of $G$ has a simple vertex. Actually, if $G$ is a non-trivial strongly chordal graph, then $G$ has at least two simple vertices [8].

The following characterization of strongly chordal graphs was proved by Farber [8].

**Proposition 7.1** (Farber [8]) A chordal graph $G$ is strongly chordal if and only if it contains no induced $k$-sun.

We next prove the main result of this section. Let $\mathcal{F}$ be the family of all the minimal forbidden induced subgraphs of the class of linear graphs, and let $F_i$ be a member of $\mathcal{F}$ which is a $P_6$-free chordal graph. We show that $F_i$ properly contains a $k$-sun ($k \geq 3$) as an induced subgraph. It is easy to see that, due to Proposition 7.1, it suffices to show both that any $P_6$-free strongly chordal graph is a linear graph, and that the $k$-sun ($k \geq 3$) is a linear graph.

The proof that a $k$-sun ($k \geq 3$) is a linear graph is given in Lemma 7.3. In order to show that a $P_6$-free strongly chordal graph $G$ is a linear graph, we will prove that $\alpha(G_A) = \lambda(G_A), \forall A \subseteq V(G)$. The proof is completed in the following four parts:

(I) we construct a strong elimination ordering $\sigma$ and a maximum independent set $I$ of $G$ with special properties,

(II) we compute a vertex coloring $\kappa$ for the graph $G$ using $\alpha(G) = |I|$ colors,

(III) we show that $\kappa$ is an optimal colinear coloring of $G$, and

(IV) we show that the equality $\lambda(G_A) = \alpha(G_A)$ holds for every induced subgraph $G_A$ of $G$.

Next, we present our proof in detail. Throughout this section we denote by $L$ the set of all the simple vertices of $G$ and by $S$ the set of all the simplicial vertices of $G$; note that $L \subseteq S$ since a simple vertex is also a simplicial vertex.

**Part (I): Construction of $I$ and $\sigma$.** Let $G$ be a $P_6$-free strongly chordal graph, and let $L$ be the set of all the simple vertices in $G$. From Definition 7.1, $G$ admits a strong elimination ordering. Using a modified version of the algorithm given by Farber in [8] we construct a strong elimination ordering $\sigma = (v_1, v_2, \ldots, v_n)$ of the graph $G$ having specific properties. Our algorithm also constructs the maximum independent set $I$ of $G$; since $G$ is a chordal graph and $\sigma$ is a perfect elimination ordering, we can use a known algorithm (e.g. see [9]) to compute a maximum independent set of the graph $G$. Throughout the algorithm, we denote by $G_i$ the subgraph of $G$ induced by the set of vertices $V(G) \setminus \{v_1, v_2, \ldots, v_{i-1}\}$, where $v_1, v_2, \ldots, v_{i-1}$ are the vertices which have already been added to the ordering $\sigma$ during the construction. Moreover, we denote by $I^*$ the set of vertices which have not been added to $\sigma$ yet and additionally do not have a neighbor already added to $\sigma$ which belongs to $I$.

In Figure 5, we present a modified version of the algorithm given by Farber [8] for constructing a strong elimination ordering $\sigma$ of $G$. Our algorithm in each iteration of Steps 3–5 adds to the ordering $\sigma$ all the vertices which are simple in $G_i$, while Farber’s algorithm selects only one simple vertex of $G_i$ and adds it to $\sigma$. We note that $L_i$ is the set of all the simple vertices of $G_i$, and also $v_k$ for $k = i$ is that vertex of $L_i$ which is added first to the ordering $\sigma$. It is easy to see that the constructed ordering $\sigma$ is a strong elimination ordering of $G$, since every vertex which is simple in $G$ is also simple in every induced subgraph of $G$. Clearly, the constructed set $I$ is a maximum independent set of $G$. 

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Input: a strongly chordal graph $G$;
Output: a strong elimination ordering $\sigma$ and a maximum independent set $I$ of $G$;

1. set $I = \emptyset$, $I^* = V(G)$, $\sigma = \emptyset$, $n = |V(G)|$, and $V_0 = V(G)$;
2. Let $(V_0, \prec_0)$ be the partial ordering on $V_0$ in which $v \prec_0 u$ if and only if $v = u$.
   set $V_1 = V(G)$ and $i = 1$;
3. Let $G_i$ be the subgraph of $G$ induced by $V_i$, that is, $V_i = V(G_i)$.
   construct an ordering on $V_i$ by $v \prec_i u$ if $v \prec_{i-1} u$ or $N_{G_i}[v] \subset N_{G_i}[u]$;
4. Let $L_i$ be the set of all the simple vertices in $G_i$.
   $k = i$;
   while $L_i \neq \emptyset$ do
     o construct an ordering on $V_k$ by $v \prec_k u$ if $v \prec_{k-1} u$ or $N_{G_k}[v] \subset N_{G_k}[u]$;
     choose a vertex $v_k$ which belongs to $L_i$ and is minimal in $(V_k, \prec_k)$, and add it to $\sigma$;
     set $V_{k+1} = V_k \setminus \{v_k\}$ and $L_i = L_i \setminus \{v_k\}$;
     o if $v_k \in I^*$ then
       set $I = I \cup \{v_k\}$ and $I^* = I^* \setminus \{v_k\}$;
       delete all neighbors of $v_k$ from $I^*$;
     o set $k = k + 1$;
   end-while;
   $i = k$;
5. if $i = n + 1$ then output the ordering $\sigma = (v_1, v_2, \ldots, v_n)$ of $V(G)$ and stop;
   else go to step 3;

Figure 5: A modified version of Farber’s algorithm for constructing a strong elimination ordering $\sigma$ and a maximum independent set $I$ of a strongly chordal graph $G$.

From the fact that $G$ is a $P_6$-free strongly chordal graph and from the construction of $I$ and $\sigma$ we obtain the following properties.

Property 7.1 Let $G$ be a $P_6$-free strongly chordal graph and let $L$ be the set of all the simple vertices of $G$. For each vertex $v_x \notin L$, there exists a chordless path of length at most 4 connecting $v_x$ to any vertex $v \in L$.

Property 7.2 Let $G$ be a $P_6$-free strongly chordal graph, let $L$ be the set of all the simple vertices of $G$, and let $I$ and $\sigma = (v_1, v_2, \ldots, v_n)$ be the maximum independent set and the ordering, respectively, constructed by our algorithm. Then,

(i) if $v_i \notin L$ and $i < j$, then $v_j \notin L$;

(ii) for each vertex $v_x \notin I$, there exists a vertex $v_i \in I$, $i < x$, such that $v_x \in N_{G_i}[v_i]$.

Next, in Part (II) we describe an algorithm for computing a vertex coloring $\kappa$ of $G$ using exactly $\alpha(G)$ colors and, then, in Part (III) we show that $\kappa$ is a colinear coloring of $G$.

Part (II): The coloring $\kappa$ of $G$. Let $G$ be a $P_6$-free strongly chordal graph, and let $L$ (resp. $S$) be the set of all the simple (resp. simplicial) vertices in $G$. We consider a maximum independent
set $I$ and a strong elimination ordering $\sigma$ of $G$, as constructed in Part (I). Now, in order to compute the colinear chromatic number $\lambda(G)$ of $G$, we compute a vertex coloring $\kappa$ of $G$ using $\alpha(G)$ colors and, then, we show that $\kappa$ is a colinear coloring of $G$. Actually, in Parts (II)-(III) we show that we can compute a colinear coloring of any $P_6$-free strongly chordal graph with $\lambda(G) = \alpha(G)$ colors, by using the constructed strong elimination ordering $\sigma$ of $G$.

First, we compute a vertex coloring $\kappa$ of $G$ using $\alpha(G)$ colors as follows:

1. Successively visit the vertices in the ordering $\sigma$ from left to right, and assign the color $\kappa(v_i)$ to the first vertex $v_i \in I$ which has not been assigned a color yet.

2. For every uncolored vertex $v_k \in N_G(v_i)$, if the collection $\{N_G[v_j] : v_j \in N_G[v_i] \text{ and } \kappa(v_j) = \kappa(v_i)\}$ is linearly ordered by inclusion, then assign the color $\kappa(v_k) = \kappa(v_i)$ to the vertex $v_k$.

3. Repeat steps 1 and 2 until there are no uncolored vertices $v_i \in I$ in $G$.

Based on this process, we obtain that every vertex $v_i$ belonging to the maximum independent set $I$ of $G$ is assigned a different color in step 1, and for each such vertex $v_i$, the collection $\{N_G[v_j] : v_j \in N_G[v_i] \text{ and } \kappa(v_j) = \kappa(v_i)\}$ is linearly ordered by inclusion. Therefore, we have assigned $\alpha(G)$ colors to the vertices of $G$. Now, if we show that there is no vertex in $\sigma$ which has not been assigned a color, then it follows that $\kappa$ is a colinear coloring of $G$ with $\alpha(G)$ colors, since by the computation of $\kappa$ the collection $\{N_G[v_j] : \kappa(v_j) = j\}$ is linearly ordered by inclusion for all $j \in \{1, 2, \ldots, \alpha(G)\}$.

The following property will be used for proving Lemma 7.1. The property holds, since simple vertices are simplicial vertices, and for every simple vertex $v \in L$ the set $\{N_G[v_x] : v_x \in N_G[v]\}$ is linearly ordered by inclusion.

**Property 7.3** For every simple vertex $v_i \in L \cap I$ of $G$, every uncolored vertex $v_x \in N_G[v_i]$ is assigned the color $\kappa(v_x) = \kappa(v_i)$ during the coloring $\kappa$ of $G$. Additionally, for each vertex $v_x \notin L$, if there exists a vertex $v \in L$ such that $v v_x \in E(G)$, then $v_x$ is assigned the color $\kappa(v_x) = \kappa(v')$ from a simple vertex $v' \in L$, $v' \leq v$.

Note that $\kappa$ is not a proper vertex coloring of $G$. Actually, since Lemma 7.1 holds, from Proposition 3.1 it follows that $\kappa$ is a proper vertex coloring of $\overline{G}$.

**Part (III): The coloring $\kappa$ is a colinear coloring of $G$.** In this part we prove the following result, by showing that there is no vertex in $\sigma$ which has not been assigned a color during the coloring $\kappa$.

**Lemma 7.1** The coloring $\kappa$ is a colinear coloring of $G$.

**Proof.** Let $G$ be a $P_6$-free strongly chordal graph, and let $L$ (resp. $S$) be the set of all the simple (resp. simplicial) vertices in $G$. We consider a maximum independent set $I$, a strong elimination ordering $\sigma$, and a coloring $\kappa$ of $G$, as computed above. Hereafter, for two vertices $v_i$ and $v_j$ in the ordering $\sigma$, we say that $v_i < v_j$ if the vertex $v_i$ appears before the vertex $v_j$ in $\sigma$.

Next, we show that there is no vertex in $\sigma$ which has not been assigned a color during the coloring $\kappa$, and since the collection $\{N_G[v_i] : \kappa(v_i) = j\}$ is linearly ordered by inclusion for all $j \in \{1, 2, \ldots, \alpha(G)\}$, from Proposition 3.2 it follows that $\kappa$ is a colinear coloring of $G$.

Assume that there exists at least one uncolored vertex $v_j$ in $G$. It follows that $v_j \notin I$, since otherwise $v_j$ would have been assigned a color in step 1 of the coloring $\kappa$. Therefore, from Property 7.2(ii) $v_j$ has a neighbor to its left in $\sigma$ which belongs to the independent set $I$. Let $v_i$ be the leftmost vertex in $\sigma$ which belongs to the independent set $I$ and did not color all its neighbors to its right in $\sigma$, and let $v_j$ be the leftmost such uncolored neighbor of $v_i$ in $\sigma$. Next, we distinguish
two cases regarding the vertex \( v_i \in I \); in the first case we consider \( v_i \) to be a simplicial vertex, i.e., \( v_i \in S \), and in the second case we consider \( v_i \notin S \). In both cases we show that our assumptions come to a contradiction.

**Case 1: The vertex \( v_i \in I \) and \( v_i \notin S \).** Since \( \sigma \) is a strong elimination ordering, each vertex \( v_i \in I \) is simple in \( G_i \) and, thus, \( \{ N_{G_i}[v_k] : v_k \in N_{G_i}[v_i] \} \) is linearly ordered by inclusion. Also, by definition, if \( v_i \in L \) then the collection \( \{ N_{G_i}[v_k] : v_k \in N_{G_i}[v_i] \} \) is linearly ordered by inclusion.

Thus, \( v_i \in I \cap S \) and \( v_i \notin L \), since otherwise \( v_j \) could have been assigned the color \( \kappa(v_j) = \kappa(v_i) \).

Therefore, there exists a neighbor \( v_k \) of \( v_i \) such that \( v_i < v_k < v_j \), \( \kappa(v_k) = \kappa(v_i) \), and neither \( N_{G_i}[v_k] \subseteq N_{G_i}[v_j] \) nor \( N_{G_i}[v_k] \supseteq N_{G_i}[v_j] \); recall that \( N_{G_i}[v_k] \subseteq N_{G_i}[v_j] \). In the case where the equality \( N_{G_i}[v_k] = N_{G_i}[v_j] \) holds, without loss of generality, we may assume that the degree of \( v_k \) in \( G \) is less than or equal to the degree of \( v_j \) in \( G \) (note that in this case \( \sigma \) is still a strong elimination ordering).

Since neither \( N_{G_i}[v_k] \subseteq N_{G_i}[v_j] \) nor \( N_{G_i}[v_k] \supseteq N_{G_i}[v_j] \), there exist vertices \( v_2 \) and \( v_3 \) in \( G \) such that \( v_2 \in N_{G_i}[v_k], v_2 \notin N_{G_i}[v_j], v_3 \in N_{G_i}[v_j], \) and \( v_3 \notin N_{G_i}[v_k] \). Since \( N_{G_i}[v_k] \subseteq N_{G_i}[v_j] \), it is easy to see that \( v_2 < v_i \) in \( \sigma \). By the assumption that \( v_i \) is the leftmost vertex in \( \sigma \) which belongs to the independent set \( I \) and has not colored all its neighbors to the right, and since \( \kappa(v_k) = \kappa(v_i) \) it follows that \( v_2 \notin I \). Thus, from Property 7.2(ii) there exists a vertex \( v_4 \in I \), such that \( v_4 < v_2 \) and \( v_2 \in N_{G_i}[v_4] \). Additionally, since \( \kappa(v_k) = \kappa(v_i) \) and \( G \) is chordal it holds that \( v_k, v_j \notin N_{G_i}[v_4] \).

Hence, the subgraph of \( G \) induced by the vertices \( \{ v_4, v_2, v_k, v_j, v_3 \} \) is a \( P_6 \). Concerning now the position of the vertex \( v_3 \) in the ordering \( \sigma \), we can have either \( v_3 < v_i \) or \( v_3 > v_i \). We will show that in both cases we come to a contradiction to our initial assumptions; that is, either it results that \( G \) has a \( P_6 \) as an induced subgraph or that the vertices should be added to \( \sigma \) in an order different than the one originally assumed.

**Case 1.1.** \( v_3 < v_i \). Assume that \( v_j \) has a neighbor \( v_3 < v_i \). Since \( v_i \) is the leftmost vertex in \( \sigma \) which belongs to the independent set \( I \) and has not colored all its neighbors to the right, it follows that \( v_3 \notin I \), since otherwise \( v_j \) would have taken the color \( \kappa(v_j) = \kappa(v_3) \) during the coloring \( \kappa \) of \( G \). Thus, similarly to the above, from Property 7.2(ii) there exists a vertex \( v_5 \in I \), such that \( v_5 < v_3 \) and \( v_3 \in N_{G_i}[v_5] \). Therefore, the vertices \( \{ v_4, v_2, v_k, v_j, v_3, v_5 \} \) induce a \( P_6 \) in \( G \), which is also chordless since \( G \) is chordal.

**Case 1.2.** \( v_3 > v_i \). Assume that \( v_j \) does not have a neighbor \( v_3 < v_i \), i.e., it has a neighbor \( v_3 > v_i \). Since \( v_i \notin L \), from Property 7.2(i) it follows that \( v_3 \notin L \). Thus, from Property 7.1 we obtain that there exists a chordless path of length at most 4 connecting \( v_3 \notin L \) to any vertex \( v \in L \). The vertex \( v_4 \) may be a simple vertex or not. However, we know that in a non-trivial strongly chordal graph there exist at least two non-adjacent simple vertices \[8\]. Thus, there exists a vertex \( v \in L, v \neq v_4 \), such that the distance \( d(v, v_3) \) of \( v_3 \) from \( v \) is at most 4, due to Property 7.1. Let \( d_m(v_3, v) = \max\{d(v_3, v) : \forall v \in L, v \neq v_4\} \). Since \( v_3 \notin L \) and \( G \) is a \( P_6 \)-free graph, it follows that \( 1 \leq d_m(v, v_3) \leq 4 \).

![Figure 6: Illustrating Case (A) and Case (B.a)](image-url)
Next, we distinguish four cases regarding the maximum distance $d_m(v_3, v)$ and show that each one comes to a contradiction. In each case we have that $\{v_3, v_2, v_k, v_j, v\}$ is a chordless path on five vertices. We first explain what is illustrated in Figures 6 and 7. Let $G_y$ be the induced subgraph of $G$, such that during the construction of $\sigma$ the vertex $v_i$ becomes simple in $G_y$, i.e., $v_i \in L_y$ and $v_y \leq v_i$. In the two figures, the vertices are placed on the horizontal dotted line in the order that appear in the ordering $\sigma$. For the vertices which are not placed on the dotted line, we are only interested about illustrating the edges among them. The vertices which are to the right of the vertical dashed line belong to the induced subgraph $G_y$ of $G$. The dashed edges illustrate edges that may or may not exist in the specific case. Next, we distinguish the four cases, and show that each one of them comes to a contradiction:

**Case (A):** $d_m(v_3, v) = 1$.

It is easy to see that $v_jv \notin E(G)$, since otherwise $v_j$ would have been assigned the color $\kappa(v)$ due to Property 7.3, and would not be an uncolored neighbor of $v_i$ as assumed. Thus, in this case there exists a $P_6$ in $G$ induced by the vertices $\{v_4, v_2, v_k, v_j, v_3, v\}$; since $G$ is a chordal graph, other edges among the vertices of this path do not exist. This is a contradiction to our assumption that $G$ is a $P_6$-free graph.

**Case (B):** $d_m(v_3, v) = 2$.

In this case there exists a vertex $v_3$ such that $\{v_3, v_5, v\}$ is a chordless path from $v_3$ to $v$. It follows that there exists a $P_7$ induced by the vertices $\{v_4, v_2, v_k, v_j, v_3, v_5, v\}$. Having assumed that $G$ is a $P_6$-free graph, the path $\{v_4, v_2, v_k, v_j, v_3\}$ is chordless and $v_j, v_k \notin N_G[v]$ due to Property 7.3, we obtain that $v_i, v_5 \in E(G)$ and $v_kv_3 \in E(G)$. Next, we distinguish three cases regarding the neighborhood of the vertex $v_3$ in $G$ and show that each one comes to a contradiction.

**(B.a)** The vertex $v_3$ does not have neighbors in $G$ other than $v_5$ and $v_j$. We will show that $v_3$ becomes simple before $v_j$ becomes simple. Assume otherwise that $v_j$ becomes simple not after $v_3$ becomes simple. Since by assumption $v_k < v_j$ we know that $v_k$ becomes simple not after $v_3$ becomes simple. Therefore, $v_k$ becomes simple not after $v_3$ becomes simple. Assume that $v_k$ becomes simple in a subgraph $G'$ of $G$. We have assumed that $v_2 < v_j < v_k$ and, thus, $v_2$ and $v_1$ have been already added to $\sigma$. It follows that $v_5$ has not been already added to $\sigma$, since it cannot become simple before at least one between $v_k$ and $v_3$ is added to $\sigma$. Therefore, when $v_k$ becomes simple in $G'$, it follows that either $N_{G'}[v_3] \subseteq N_{G}[v_j]$ or $N_{G'}[v_5] \supseteq N_{G}[v_j]$. Therefore, since we have assumed that $v_3$ does not have neighbors in $G$ other than $v_5$ and $v_j$, it follows that $v_3$ becomes also simple in $G'$, along with $v_k$. However, $v_j$ is not simplicial in $G'$ since $v_i, v_3 \notin E(G)$ and, thus, $v_j$ is not simple in $G'$. Therefore, $v_3$ will be added to $\sigma$ before $v_j$ will be added to $\sigma$.

Additionally, since $v_5$ sees the simple vertex $v$, from Property 7.3 it follows that $v_5$ will be assigned a color from a simple vertex and $v_5 \notin I$. Moreover, by assumption $v_j \notin I$. Therefore, $v_3 \in I$ and since we have showed that $v_3 < v_j$, it follows that $v_j$ is the only uncolored neighbor of $v_3$ to its right in $\sigma$ and, thus, $v_j$ will be assigned the color $\kappa(v_j) = \kappa(v_3)$. This is a contradiction to our assumption that $v_j$ has not been assigned a color.

So far, we have shown that if the vertex $v_3$ does not have neighbors in $G$ other than $v_5$ and $v_j$, then we come to a contradiction to our assumptions. Since we initially assumed that $v_3 > v_i$ in $\sigma$, i.e., that $v_3$ does not become simple before $v_i$ becomes simple, we continue by examining the cases where $v_3$ has neighbors in $G_y$ other than $v_5$ and $v_j$.
Case (B.b) The vertex \( v_3 \) has two neighbors \( v_5 \) and \( v'_5 \) in \( G_y \), such that \( v_5v'_5 \notin E(G) \). Since we have assumed that the maximum distance of the vertex \( v_3 \) from \( v \) in \( G \), for any vertex \( v \in L \), \( v \neq v_4 \), is \( d_m(v_3, v) = 2 \), and \( v_3 \) has no neighbor belonging to \( L \) since \( G \) is a \( P_6 \)-free graph, it follows that \( v_5, v'_5 \notin L \) and there exist vertices \( v, v' \in L \) such that the vertices \( \{v_3, v_5, v\} \) induce a chordless path from \( v_3 \) to \( v \) and \( \{v_3, v'_5, v'\} \) induce a chordless path from \( v_3 \) to \( v' \). It is easy to see that \( v \neq v' \) and \( vv' \notin E(G) \) since \( G \) is a chordal graph. Therefore, from Case (B.a) we have \( v_k, v_j \in N_G[v_5] \) and \( v_k, v_j \in N_G[v'_5] \). However, in this case there exists a \( C_4 \) in \( G \) induced by the vertices \( \{v_5, v_3, v'_5, v_k\} \), since by assumption \( v_5v'_5 \notin E(G) \) and \( v_3v_k \notin E(G) \). Concluding, the vertex \( v_3 \) cannot have two neighbors \( v_5 \) and \( v'_5 \) in \( G \), such that \( v_5v'_5 \notin E(G) \). Thus, \( v_3 \in S \).

Case (B.c) The vertex \( v_3 \) has two neighbors \( v_5 \) and \( v'_5 \) (where \( v_5 \neq v_j \) and \( v'_5 \neq v_j \)) in \( G_y \), such that \( v_5v'_5 \in E(G) \), but neither \( N_{G_y}[v_5] \subseteq N_{G_y}[v'_5] \) nor \( N_{G_y}[v'_5] \subseteq N_{G_y}[v_5] \); thus, there exist vertices \( v_6 \) and \( v'_6 \) in \( G_y \) such that \( v_5v_6 \in E(G) \) and \( v_5v'_6 \notin E(G) \) and also, \( v'_5v'_6 \in E(G) \) and \( v'_5v_6 \notin E(G) \). Since \( v_3 \in S \), it follows that \( v_6, v'_6 \notin N_G[v_3] \). Since \( d_m(v_3, v) = 2 \), there exists a vertex \( v \in L \) such that \( \{v_3, v, v_6\} \) is a chordless path from \( v_3 \) to \( v \). Similarly, there exists a vertex \( v' \in L \) such that \( \{v_3, v, v'_6\} \) is a chordless path from \( v_3 \) to \( v' \). We have that \( v \neq v' \), \( v_6 \notin E(G) \) and \( v'_5v_6 \notin E(G) \), since otherwise \( v \) and \( v' \) would not be simple in \( G \). Additionally, \( vv' \notin E(G) \), \( vv'_6 \notin E(G) \), and \( vv'_5 \notin E(G) \), since \( G \) is a chordal graph. Therefore, from Case (B.a) we have \( v_k, v_j \in N_G[v_5] \) and \( v_6, v'_6 \in N_{G_y}[v'_5] \). Assume that there exist vertices \( v'', v''' \in L \), such that \( v_6v''' \in E(G) \) and \( v'_6v'' \in E(G) \). It is easy to see that at least one of the equivalences \( v \equiv v''' \) and \( v' \equiv v'' \) holds, otherwise \( G \) has a \( P_6 \) induced by the vertices \( \{v''', v_6, v_5, v'_5, v'_6, v''\} \). Without loss of generality, assume that \( v \equiv v''' \) holds.

Since \( v \in L \), \( v_5, v_6 \in N_G[v], v'_5 \in N_G[v_5] \), and \( v'_6 \notin N_G[v_6] \), it follows that \( N_G[v_6] \subset N_G[v_5] \). In the case where \( v_k, v_j \notin N_G[v_6] \) we have \( v_6 \in L \) and, thus, \( v_6 \) would be added to \( \sigma \) in the first iteration which is a contradiction to our assumption that \( v_6 \in G_y \). Assume that \( v_3v_6 \in E(G) \); it follows that \( v_kv_6 \in E(G) \), since otherwise \( G \) has a \( P_6 \) induced by the vertices \( \{v_4, v_2, v_k, v_3, v_6, v\} \). If \( v' \equiv v'' \), the same arguments hold for \( v'_6 \) too and, thus, if \( v_jv'_6 \in E(G) \) then \( v_3v'_6 \in E(G) \). In the case where \( v' \neq v'' \) we have \( v'_6v_k \in E(G) \), since otherwise \( G \) has a \( P_6 \) induced by the vertices \( \{v_4, v_2, v_k, v'_5, v'_6, v''\} \). Thus, in any case \( v_6, v'_6 \in N_G[v], v'_5 \in N_{G_y}[v_5] \), and \( G \) has a \( 3 \)-sun induced by the vertices \( \{v_k, v_5, v'_5, v'_6, v_6, v_3\} \). Since other edges between the vertices of the \( 3 \)-sun do not exist, it follows that at least one of the vertices \( v_6 \) and \( v'_6 \) does not belong to the neighborhood of \( v_k \) and, thus, of \( v_j \) in \( G \). Without loss of generality, let \( v_6 \) be that
The vertex. Thus, \( v_6 \in L \) and, subsequently, \( v_6 \) will be added to \( \sigma \) during the first iteration. Thus, \( v_3 \) is simple and will be added to \( \sigma \) during the second iteration, along with \( v_2 \), while \( v_i \) will be added to \( \sigma \) after the second iteration (i.e., \( v_3 < v_y \leq v_i \)). This is a contradiction to our assumption that \( v_3 > v_i \).

Using similar arguments, we can prove that \( v_3 \) will be added to \( \sigma \) before \( v_i \), even if there exist edges between \( v_2 \) and the vertices \( v_5, v_6, v_0, \) and \( v_0' \). Actually, it easily follows that \( v_2v_6 \notin E(G) \), since \( v_0v_5 \notin E(G) \) and \( G \) is a chordal graph. Additionally, \( v_2v_5 \notin E(G) \), since we know that \( v_2v_6 \notin E(G) \), \( v_2v_3 \notin E(G) \) and \( v_2 \) is simple in \( G_2 \). Therefore, whether \( v_2v_6 \in E(G) \) or not, it does not change the fact that \( v_3 \) becomes simple after the first iteration and, thus, \( v_3 \) is added to \( \sigma \) before \( v_i \). Note, even in the case where \( v \equiv v_4 \) or \( v' \equiv v_4 \) (in the case where \( v_4 \in L \)), it similarly follows that \( v_6 \in L \) or \( v_6 \notin L \) respectively and, thus, \( v_3 \) becomes simple after the first iteration and is added to \( \sigma \) before \( v_i \).

**Case (C):** \( d_m(v_3, v) = 3 \).

In this case there exist vertices \( v_5 \) and \( v_6 \) such that \( \{v_3, v_5, v_6, v\} \) is a chordless path from \( v_3 \) to \( v \). Since now \( G \) has a \( P_6 \), it follows that \( v_5v_7 \in E(G) \) and, additionally, some other edges must exist among the vertices \( v_2, v_k, v_j, v_5, \) and \( v_6 \). In any case, we will prove that either \( N_G[v_6] \subseteq N_G[v_5] \) or \( N_G[v_6] \subseteq N_G[v_5] \) and, thus, \( v_3 \in L \). Similarly to Case (B), we distinguish three cases regarding the neighborhood of the vertex \( v_3 \) in \( G \) and show that if \( v_3 \notin L \) then each one comes to a contradiction.

**(C.a)** The vertex \( v_3 \) does not have neighbors in \( G \) other than \( v_5 \) and \( v_j \). Since \( v_5v_j \in E(G) \) then some other edges must exist, since otherwise \( G \) has a \( P_7 \) induced by the vertices \( \{v_4, v_2, v_k, v_j, v_5, v_6, v\} \).

- Consider the case where \( v_2v_5 \in E(G) \). Then either \( v_2v_5 \in E(G) \) or \( v_kv_6 \in E(G) \), since \( G \) has a \( P_6 \). In the case where \( v_2v_5 \in E(G) \) then \( v_5v_7 \notin E(G) \). In the case where \( v_kv_6 \in E(G) \) then either \( v_5v_7 \in E(G) \) or \( v_kv_6 \notin E(G) \). For both cases, assume that \( v_5v_7 \in E(G) \). Since \( v_5 \) and \( v_j \) are adjacent to \( v_k \) and \( v_3 \), and \( v_5v_6 \notin E(G) \), it follows that \( v_5 \) and \( v_3 \) cannot be added to \( \sigma \) unless at least one of \( v_k \) and \( v_3 \) is added to \( \sigma \). Additionally, by assumption, \( v_i < v_k < v_j \) and \( v_i < v_3 \). Thus, when \( v_i \) is added to \( \sigma \) it follows that \( v_k \) and \( v_3 \) have not been added to \( \sigma \) yet and, thus, \( v_5 \) and \( v_j \) have not been added to \( \sigma \) yet neither, i.e., \( v_k, v_3, v_5, \) and \( v_i \) belong to \( G_i \). Thus, \( N_{G_i}[v_5] \supseteq N_{G_i}[v_3] \).

If \( v_2v_5 \in E(G) \) then \( N_{G_2}[v_5] \supseteq N_{G_2}[v_6] \). Then, since in Case 1.2 we have assumed that there exists no vertex \( v_x \) such that \( v_x \notin v_5, v_xv_6 \in E(G) \), and \( v_xv_k \notin E(G) \); thus, for every neighbor \( v_x \) of \( v_j \) such that \( v_2 < v_x < v_5 \), it follows that \( v_xv_5 \in E(G) \). Also, there exists no vertex \( v_x \) such that \( v_x < v_2 \) and \( v_xv_j \in E(G) \). Indeed, if we assume otherwise then \( v_xv_5 \in E(G) \), and since \( v_2v_6 \in E(G) \), it follows from Definition 7.1 that \( v_2v_6 \in E(G) \). This is a contradiction on the choice of \( v_2 \).

Summarizing, there exists no vertex \( v_x \) such that \( v_x < v_5, v_xv_6 \in E(G) \), and \( v_xv_6 \notin E(G) \). Therefore, since \( N_{G_i}[v_5] \supseteq N_{G_i}[v_3] \), it follows that \( N_{G_i}[v_3] \supseteq N_{G_i}[v_5] \). Thus, \( v_3 \in L \) which is a contradiction.

Consider now the case where \( v_2v_5 \notin E(G) \). In other words, \( v_5 \) does not have a neighbor \( v_x \) in \( \sigma \) such that \( v_x < v_i, v_xv_k \in E(G) \) and \( v_xv_j \notin E(G) \). Then \( v_5v_6 \in E(G) \) and either \( v_5v_6 \in E(G) \) or \( v_5v_6 \notin E(G) \). We now show that in both cases \( v_5v_6 \in E(G) \). If \( v_5v_6 \in E(G) \) then either \( v_6 < v_i \) or \( v_6v_j \in E(G) \), since \( N_{G_i}[v_6] \subseteq N_{G_i}[v_j] \). However, even if \( v_6 < v_i \) then again \( v_6v_j \in E(G) \), since we have
proved that $v_3$ does not have a neighbor $v_x$ in $\sigma$ such that $v_x < v_i$, $v_x v_k \in E(G)$ and $v_x v_j \notin E(G)$. Also, if $v_1v_6 \in E(G)$ then again $v_6v_j \in E(G)$, since $v_i \in S$. Therefore, in both cases $v_6v_j \in E(G)$. We now show that $v_5$ does not have a neighbor $v_x$ such that $v_x < v_i$ and $v_xv_j \notin E(G)$. Indeed, if such a vertex $v_x$ exists then also $v_x v_k \notin E(G)$. If $v_x \in L$ then $\kappa(v_x) = \kappa(v)$; thus, we can prove similarly to Case (B.a) that $v_3 < v_j$ and $\kappa(v_j) = \kappa(v_3)$. In the case where $v_x \notin L$, then $d(v_x, v_j) \geq 1$ for any vertex $v_x \in L$; in this case it follows that $G$ has a $P_6$ induced by the vertices $\{v_4, v_2, v_5, v_6, v_x, v_j\}$, since $v_x v_k \notin E(G)$. Therefore, we have showed that if $v_5$ does not have a neighbor $v_x$ such that $v_x < v_i$ and $v_xv_j \notin E(G)$. Assume that $v_5$ has a neighbor $v_x$ such that $v_x > v_3$ and $v_xv_j \notin E(G)$. Then $v_6v_k \notin E(G)$ since $N_G[v_k] \subseteq N_G[v_j]$. Similarly to the above it follows that if $v_3$ has such a neighbor $v_x$ then either $v_x \in L$ or $G$ has a $P_6$. Therefore, we have showed that $N_G[v_3] \subseteq N_G[v_j]$ and, thus, $v_3 \in L$ which is a contradiction.

- Consider now the case where $v_3v_5 \notin E(G)$. Then $v_3, v_j \in N_G[v_3]$, since otherwise $G$ has a $P_6$. Assume that $v_3$ has a neighbor $v_x$, such that $v_xv_j \notin E(G)$. Since $v_xv_5 \notin E(G)$, it follows similarly to the above that in this case $G$ has a $P_6$. Therefore, we have showed that $N_G[v_3] \subseteq N_G[v_j]$ and, thus, $v_3 \in L$ which is a contradiction.

(C.b) The vertex $v_3$ has two neighbors $v_5$ and $v'_5$ in $G_y$, such that $v_3v'_5 \notin E(G)$. Using the same arguments as in Case (B.b), we obtain that in this case $G$ has a $C_4$ which is a contradiction to our assumptions.

(C.c) The vertex $v_3$ has two neighbors $v_5$ and $v'_5$ (where $v_5 \neq v_j$ and $v'_5 \neq v_j$) in $G_y$, such that $v_3v'_5 \in E(G)$, and neither $N_G[v_5] \subseteq N_G[v'_5]$ nor $N_G[v'_5] \subseteq N_G[v_5]$; that is, there exist vertices $v_6$ and $v'_6$ in $G_y$ such that $v_3v_6 \in E(G)$ and $v_3v'_6 \notin E(G)$ and, also, $v_3v'_6 \in E(G)$ and $v'_3v_6 \notin E(G)$. Similarly to Case (B.c), we can prove that this case comes to a contradiction as well. Note that, in this case $d_m(v_3, v) = 3$ and, thus, there exists a chordless path $\{v_3, v_5, v_7, v\}$ from $v_3$ to $v$. Again, at least one of $v \equiv v''$ and $v' \equiv v''$ must hold, since otherwise $G$ has a $P_6$ induced by the vertices $\{v'', v_3, v_5, v'_3, v'_6, v''\}$. Using the same arguments as in Case (B.c), we obtain that if $v \equiv v''$ then $v_3, v_j \notin N_G[v_6]$. However, now, we must additionally have $v_3v_7 \notin E(G)$, since otherwise $G$ has a $C_4$ induced by the vertices $\{v, v_7, v_5, v_6\}$. Therefore, as in Case (B.c) we obtain $v_6 \in L$, which is a contradiction to our assumption that the vertex $v_3$ appears in the ordering before the vertices $v_6, v'_6, v_5$, and $v'_5$.

**Case (D):** $d_m(v_3, v) = 4$.

In this case there exist vertices $v_5, v_6$ and $v_7$ such that $\{v_3, v_5, v_6, v_7, v\}$ is a chordless path from $v_3$ to $v$. Since now $G$ has a $P_9$, it follows that $v_5v_j \in E(G)$ and, additionally, some other edges must exist. Similarly to Cases (A) and (B), we distinguish three cases regarding the neighborhood of the vertex $v_3$ in $G$ and show that if $v_3 \notin L$ then each one comes to a contradiction.

(D.a) The vertex $v_3$ does not have neighbors in $G$ other than $v_5$ and $v_j$. If we assume that $v_3 \notin L$, then $v_5$ has a neighbor in $G$ which is not a neighbor of $v_j$ and, additionally, $v_j$ has a neighbor in $G$ which is not a neighbor of $v_5$. Thus, we can have one of the following three cases, each of which comes to a contradiction:

- $v_2 \in N_G[v_5]$ and $v_7 \in N_G[v_j]$. Now, we have that $v_2v_6 \in E(G)$, since otherwise $G$ has a $P_6$ induced by the vertices $\{v_4, v_2, v_5, v_6, v_7, v\}$. However, in this case $v_2$ would not be simple in $G_2$, where $G_2$ is the subgraph of $G$ induced by the vertices to the right of $v_2$ in $\sigma$, since $v_7 \in N_G[v_5]$ and $v_7 \notin N_G[v_5]$ and, also, $v_3 \notin N_G[v_5]$.
and \( v_3 \notin N_G[v_6] \). Indeed, it suffices to show that the vertices \( v_5, v_6, v_7, \) and \( v_3 \) belong to the induced subgraph \( G_2 \) of \( G \).

We know that \( v_5, v_3 \in N_G[v_j] \), and thus, \( v_5 > v_i \) and \( v_3 > v_i \), since we have assumed that \( v_i \) does not have a neighbor \( v_x \), such that \( v_x < v_i \). Additionally, from \( v_7 \in N_G[v_j] \) it follows that \( v_6 \notin N_G[v_j] \), since otherwise \( G \) has a \( C_4 \) induced by the vertices \( \{v_j, v_5, v_6, v_7\} \). Therefore, \( v_6, v_7 \in N_G[v_j] \), and thus, \( v_i < v_6 \) and \( v_i < v_7 \). Therefore, the vertices \( v_5, v_6, v_7, \) and \( v_3 \) belong to the induced subgraph \( G_2 \) of \( G \), and thus, the vertex \( v_2 \) is not simple in \( G_2 \), which is a contradiction to our assumption that \( \sigma \) is a strong elimination ordering.

- \( v_k \notin N_G[v_5] \) and \( v_6 \notin N_G[v_j] \). From \( v_k \notin N_G[v_5] \) we obtain that \( v_2, v_i \notin N_G[v_5] \). In this case \( G \) has a \( P_k \) induced by the vertices \( \{v_4, v_2, v_k, v_5, v_6, v_7\} \). This path is chordless since \( G \) is a chordal graph.

- \( v_i \notin N_G[v_5] \) and \( v_6 \notin N_G[v_j] \). In this case, we have a \( P_k \) in \( G \) induced by the vertices \( \{v_4, v_2, v_k, v_5, v_6, v_7, v\} \), and \( v_i \notin N_G[v_5] \) we obtain that \( v_2 \notin N_G[v_5] \) and, thus, \( v_6v_k \in E(G) \). Now, \( G \) has a \( 3 \)-sun induced by the vertices \( \{v_5, v_6, v_j, v_6, v_4, v_3\} \), since we have assumed that \( v_i \notin E(G), v_4v_j \notin E(G) \), other edges do not exist by assumption. This is a contradiction to our assumption that \( G \) is a strongly chordal graph.

Using similar arguments as in Case (B.a) and Case (C.a), we can prove that either \( N_G[v_5] \subseteq N_G[v_j] \) or \( N_G[v_6] \subseteq N_G[v_j] \), and thus, \( v_3 \in L \). Similarly to Cases (B) and (C), we distinguish three cases regarding the neighborhood of the vertex \( v_3 \) in \( G \) and can show that each one comes to a contradiction.

(D.b) The vertex \( v_3 \) has two neighbors \( v_5 \) and \( v_3' \) in \( G_y \), such that \( v_3v_5 \notin E(G) \). Using the same arguments as in Case (B.b), we obtain that in this case \( G \) has a \( C_4 \) which is a contradiction to our assumptions.

(D.c) The vertex \( v_3 \) has two neighbors \( v_5 \) and \( v_3' \) (where \( v_5 \neq v_j \) and \( v_3' \neq v_j \)) in \( G_y \), such that \( v_5v_3' \in E(G) \), and neither \( N_{G_y}[v_5] \subseteq N_{G_y}[v_3'] \) nor \( N_{G_y}[v_3'] \subseteq N_{G_y}[v_5] \). Using the same arguments as in Cases (B.c) and (C.c), we can prove that this case comes to a contradiction.

**Case 2:** The vertex \( v_i \in I \) and \( v_i \notin S \). Since \( \sigma \) is a strong elimination ordering, each vertex \( v_i \in I \) is simple in \( G_i \) and, thus, \( \{N_{G_i}[v_i]: v_i \in N_{G_i}[v_i]\} \) is linearly ordered by inclusion. Since \( v_i \) is not a simplicial vertex in \( G \), there exist at least two vertices \( v_1, v_2 \in N_{G}(v_i) \) such that \( v_1v_2 \notin E(G) \) and \( v_1 < v_i < v_2 \). If there exists a neighbor \( v_k \) of \( v_i \) such that \( v_i < v_k < v_j \) and neither \( N_{G}[v_k] \subseteq N_{G}[v_j] \) nor \( N_{G}[v_j] \subseteq N_{G}[v_k] \), then as we showed in Case 1, we come to a contradiction; recall that we have assumed that \( v_j \) is the uncolored vertex.

Assume that such a vertex \( v_k \) does not exist. Therefore, since \( \kappa(v_j) \neq \kappa(v_i) \) it follows that neither \( N_{G}[v_i] \subseteq N_{G}[v_j] \) nor \( N_{G}[v_j] \subseteq N_{G}[v_i] \), and, thus, there exists a vertex \( v_2 \) such that \( v_2 < v_i < v_j \), \( v_2v_i \in E(G) \), and \( v_2v_j \notin E(G) \). Additionally, there exists a vertex \( v_3 \in \sigma \) such that \( v_3v_i \notin E(G) \) and \( v_3v_j \in E(G) \). Thus, \( \{v_2, v_i, v_3, v_j\} \) is a chordless path on 4 vertices. Additionally, since \( v_2 \) is a neighbor of \( v_i \) in \( I \) it follows that \( v_2 \notin I \), and from Property 7.2(i) it follows that there exists a vertex \( v_4 \in I \) in \( \sigma \) such that \( v_4 < v_2 \) and \( v_4v_2 \in E(G) \). Therefore, \( \{v_4, v_2, v_i, v_3, v_j\} \) is a chordless path on 5 vertices. Using the same arguments as in Case 1, we can come to a contradiction by substituting \( v_k \) by \( v_i \) in the proof of Case 1.

From Cases 1 and 2 we conclude that using the constructed strong elimination ordering \( \sigma \) of \( G \), we have proved that there is no uncolored vertex in \( \sigma \), and since the set \( \{N_{G}[v_k]: \kappa(v_k) = j\} \) is linearly ordered by inclusion for every \( j \in \{1, 2, \ldots, \alpha(G)\} \), it follows that \( \kappa \) is a colinear coloring of \( G \). Thus, the lemma holds.

\[ \square \]
Part (IV): The equality $\lambda(G_A) = \alpha(G_A)$ holds for every $A \subseteq V(G)$. It is easy to see that, in Parts (I)–(III), we have showed that we can assign a colinear coloring with $\lambda(G) = \alpha(G)$ colors to any $P_6$-free strongly chordal graph, by using the constructed strong elimination ordering $\sigma$ of $G$.

From Corollary 3.1, we have that $\lambda(G) \geq \alpha(G)$ holds for any graph $G$. Since $\kappa$ is a colinear coloring of $G$ using $\alpha(G)$ colors, it follows that the equality $\lambda(G) = \alpha(G)$ holds for $G$. Since every induced subgraph of a strongly chordal graph is strongly chordal \cite{8}, we can construct a strong elimination ordering $\sigma$ as described above for every induced subgraph $G_A$ of $G$, $\forall A \subseteq V(G)$; thus, we can assign a coloring $\kappa$ to $G_A$ with $\alpha(G_A)$ colors. Concluding, the equality $\lambda(G_A) = \alpha(G_A)$ holds for every induced subgraph $G_A$ of a strongly chordal graph $G$ and, therefore, any strongly chordal graph $G$ is a linear graph.

Therefore, in Parts (I)–(IV) we have proved the following result.

**Lemma 7.2** Any $P_6$-free strongly chordal graph is a linear graph.

From Lemma 7.2, we obtain the following result.

**Lemma 7.3** If $G$ is a $k$-sun graph ($k \geq 3$), then $G$ is a linear graph.

**Proof.** Let $G$ be a $k$-sun graph. It is easy to see that the equality $\alpha(G) = \lambda(G)$ holds for the $k$-sun $G$. Since a $k$-sun constitutes a minimal forbidden subgraph for the class of strongly chordal graphs, it follows that every induced subgraph of a $k$-sun is a strongly chordal graph and, thus, from Lemma 7.2 we obtain that $G$ is a linear graph. \Halmos

From Lemmas 7.2 and 7.3, we also derive the following results.

**Proposition 7.2** Linear graphs form a superclass of the class of $P_6$-free strongly chordal graphs.

We have proved that any $P_6$-free chordal graph which is not a linear graph has a $k$-sun as an induced subgraph; however, the $k$-sun itself is a linear graph. The interest of these results lies on the following characterization that we obtain for the class of linear graphs in terms of forbidden induced subgraphs.

**Theorem 7.1** Let $\mathcal{F}$ be the family of all the minimal forbidden induced subgraphs of the class of linear graphs, and let $F_i$ be a member of $\mathcal{F}$. The graph $F_i$ is either a $C_n$ ($n \geq 4$), or a $P_6$, or it properly contains a $k$-sun ($k \geq 3$) as an induced subgraph.

In light of the above result, it would be interesting to investigate whether or not linear graphs are characterized completely by a finite set of forbidden induced subgraphs. To this end, we need to investigate the $P_6$-free chordal graphs which are forbidden subgraphs for linear graphs; as we have shown these graphs properly contain a $k$-sun. An example of such a graph is the complement of the graph depicted in Figure 2; this graph is a $P_6$-free chordal graph on 9 vertices which properly contains a 4-sun, and is not a linear graph.

In general, an example of a forbidden induced subgraph of linear graphs is a graph $H$ on $2k + 1$ vertices which properly contains a $k$-sun $S_k$ ($k \geq 4$) such that $H = \{v\} \cup \{u_1, u_2, \ldots, u_k\} \cup \{w_1, w_2, \ldots, w_k\}$ and $v$ is adjacent to every vertex of the clique $W = \{w_1, w_2, \ldots, w_k\}$ and to exactly two vertices, say, $u_i$ and $u_j$ ($j < i$) of the independent set $U = \{u_1, u_2, \ldots, u_k\}$ such that $i \neq j + 1 \pmod{k}$; recall that $U$ is the independent set and $W$ is the clique of the sun $S_k$. We claim that the $P_6$-free chordal graphs which are forbidden subgraphs for linear graphs do not restrict to graphs with such a structure and, also, that linear graphs are characterized completely by a finite set of forbidden induced subgraphs.
A finite set of forbidden subgraphs could lead to a recognition algorithm for linear graphs. Such an algorithm would require the detection of graphs of a specific structure which properly contain a $k$-sun. It is worth noting that finding a $k$-sun in a general graph has been recently proved to be NP-complete [10]. However, one can answer the question whether or not a chordal graph $G$ contains a $k$-sun by using Farber’s algorithm [8]; if $G$ contains a $k$-sun as an induced subgraph, Farber’s algorithm reports that $G$ is not a strongly chordal graph and, also, returns an induced subgraph of $G$ which contains a $k$-sun. However, there is no known polynomial algorithm for detecting and reporting a $k$-sun in a chordal graph.

Investigating an algorithm for detecting and reporting a $k$-sun in a chordal graph is of great interest, since it could be a step toward the recognition of the class of linear graphs. Additionally, such an algorithm along with a minimal set of forbidden induced subgraphs could help us to characterize and provide properties of linear graphs which could be used for finding polynomial solutions for problems on linear graphs, which are NP-complete on chordal graphs.

8 Concluding Remarks

In this paper we introduced the colinear coloring on graphs, and proposed a colinear coloring algorithm that can be applied to any graph $G$. Based on the colinear coloring we defined two graph properties, namely the $\chi$-colinear and $\alpha$-colinear properties, and characterized known graph classes in terms of these properties. We also studied the graphs that are characterized completely by the $\chi$-colinear or the $\alpha$-colinear property, which form two new classes of perfect graphs, namely colinear and linear graphs.

An interesting question would be to study structural and recognition properties of colinear and linear graphs and see whether they can be characterized by a finite set of forbidden induced subgraphs. Moreover, an obvious though interesting open question would be whether combinatorial and/or optimization problems can be efficiently solved on the classes of linear and colinear graphs.

In addition, it would be interesting to study the relation between the colinear chromatic number and other coloring numbers such as the harmonious number and the achromatic number on classes of graphs.

References


