FREQUENCY SPECTRUM OF THE BISPHERICAL HOLLOW SYSTEM: THE CASE OF THE NONUNIFORM THICKNESS HUMAN SKULL

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Abstract

In the present work we discuss the role of the thickness nonuniformities of the human dry skull on its frequency spectrum. The mathematical modelling is based on the three-dimensional theory of elasticity, the approximation of the skull by an isotropic material occupying the region bounded by two non-concentric spheres and the introduction of the bispHERical coordinates system. The mathematical analysis is based on the construction of the Navier eigenfunctions of the problem and the representation of the solution in terms of them. The frequency equation is solved numerically and the role of various parameters entering the problem is extensively discussed.

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1. Introduction

The complexity of the human cranial system presents an extremely difficult task to one wishing to perform detailed analysis of the physical processes of the human head by mathematical modelling. For this reason geometrical and material approximations are typically used for analytical investigations [1,2]. The human skull is a complex structure made up of several bones each with its own unique internal and external geometry. In cross section, the skull varies in thickness from about 3/8 to 1/2 in, increasing in this dimension towards the base of the skull. It consists of an inner and outer table of compact bone with a highly vascular middle layer resembling a honeycomb, called the diploë, and appears to be transversely isotropic. In analytical investigations geometrical approximations and mainly the geometry of spherical and prolate spheroidal shells have been used [3,4,5]. In a recent paper [6] the dynamic characteristics of the human dry skull were presented and the results obtained were favourably compared to the experimental ones [7]. The analysis in Ref. 6 was based on the three-dimensional theory of elasticity and the representation of the displacement field in terms of the Navier eigenvectors. The role of a small deviation of the spherical to spheroidal geometry on the frequency spectrum of the human dry skull was presented very recently in [8]. In Ref. 8 the analysis was based on the three-dimensional theory of elasticity, complex analysis techniques and the constructed for this problem Navier eigenvectors. The conclusion from the results obtained in Ref. 8 is that the deviation of the spherical geometry to spheroidal one with sufficiently small eccentricity does not appreciably influence the frequency spectrum of the human dry skull.

In the present work we deal with the role of the thickness nonuniformities, due to the nature and the existence of the facial bones, of the human skull on its frequency spectrum. The mathematical modelling is based on the three-dimensional theory of elasticity and the approximation of the skull by an isotropic material occupying the region bounded by two non-concentric spheres. The bispHERical coordinates system
was used to describe the geometry introduced by the skull morphology. We note that
the analysis we present is only valid for small deviations of the system from the one of
the two concentric spheres. The mathematical analysis is based on the construction of
the Navier eigenfunctions for the problem under discussion and the representation of its
solution in terms of this complete set of vector functions in the space of solutions of
time-independent Navier equation. The frequency equation is constructed by imposing
the satisfaction of the boundary conditions and it is solved numerically. From the
results obtained we discover the role of the various parameters entering the problem
(especially of the thickness nonuniformities) on the frequency spectrum of the human
dry skull.

2. The Bispherical System

We consider the region \( V \) bounded by two non-concentric spheres. The bispherical
coordinate system is the most suitable to describe the geometry introduced by the two
non-concentric spheres, \( S_1 \) and \( S_2 \) (Fig. 1).
Figure 1: Two Non-concentric Spheres System.

Every point in $\mathbb{R}^3$ is determined by specific values of the bispherical coordinates $(\vartheta, j, \varphi)$ [9] which are related to the cartesian ones by the following relations (assuming that the $z$-axis is the rotation axis of the system):

$$x = \frac{\alpha \sin \vartheta \cos \varphi}{\cosh j - \cos \vartheta}$$  \hspace{1cm} (1)

$$y = \frac{\alpha \sin \vartheta \sin \varphi}{\cosh j - \cos \vartheta}$$  \hspace{1cm} (2)

$$z = \frac{\alpha \sinh j}{\cosh j - \cos \vartheta}.$$  \hspace{1cm} (3)

It is well-known for the bispherical system of coordinates that the surfaces of constant $j$ are spheres and the surfaces of constant $\vartheta$ are either apple-shaped surfaces with
dimples on the $x$-axis or spindles. Surfaces of constant $\varphi$ are half planes as in toroidal coordinates.

The variable $j$ has the range from $-\infty$ to $\infty$. Positive values of $j$ are associated with spheres whose centers lay above the $xy$-plane and negative values with spheres with centers below the $xy$-plane. In both cases, the center of the sphere is $a \coth j$ and the radius is $a / |\sinh j|$. Note that $j = 0$ corresponds to the $xy$-plane (sphere of infinite radius) while the values $j = -\infty, +\infty$ correspond to the points $(0,0,-\alpha)$ and $(0,0,\alpha)$, respectively. That means, as $j$ increases in absolute value, the spheres shrink, changing their centers, tending to coincide with points $(0,0,-\alpha)$ or $(0,0,\alpha)$ depending on the sign of $j$.

The variable $\vartheta$ ranges from 0 to $\pi$. The surfaces $\vartheta = \text{const.}$ are apple-shaped if $\vartheta < \pi / 2$ and spindle-shaped if $\vartheta > \pi / 2$. The case $\vartheta = \pi / 2$ corresponds to a sphere of radius $\alpha$; for the case $\vartheta = \pi$ we have a line segment of length $2\alpha$ connecting the foci $(0,0,\alpha)$, $(0,0,-\alpha)$. The case $\vartheta = 0$ corresponds to infinite pieces of $z$-axis, which are complements of the segment connecting the foci.

Before stating the problem we have to determine the parameter $\alpha$ as well as the values $j_1$ and $j_2$ corresponding to the surfaces $S_1$ and $S_2$, respectively, of the system under consideration. Given the radii of the two spheres $r_1$ and $r_2$ and the eccentricity $e$, we have:

$$\frac{\alpha}{\sinh j_1} = r_1$$  \hspace{1cm} (4)

$$\frac{\alpha}{\sinh j_2} = r_2$$  \hspace{1cm} (5)

$$\alpha \coth j_1 - \alpha \coth j_2 = e.$$  \hspace{1cm} (6)
The solution of the system of equations (4), (5) and (6) gives that:

\[
\alpha = \frac{\sqrt{(r_1^2 - r_2^2 - \varepsilon^2)^2 - 4\varepsilon^2 r_2^2}}{2\varepsilon}
\]

(7)

\[
e^{-i\mu} = \sqrt{\left(\frac{\mu}{r_1}\right)^2 + 1 - \frac{\mu}{r_1}}
\]

(8)

\[
e^{-i\nu} = \sqrt{\left(\frac{\nu}{r_2}\right)^2 + 1 - \frac{\nu}{r_2}}
\]

(9)

The parameters \(\alpha, \mu, \nu\) describe uniquely the system of two spheres with given eccentricity \(\varepsilon\) and radii \(r_1, r_2\).

The actual region \(V\), occupied by the elastic material corresponds to the domain

\[
V = \{(\vartheta, j, \varphi) : j \in (j_1, j_2), \ \vartheta \in [0, \pi], \ \varphi \in [0, 2\pi]\}
\]

and forms a bispherical shell.

3. Analysis of the Harmonic Motion

Let us consider the system under consideration executes harmonic motion. The displacement field \(u(r)\) in the skull region satisfies, after suppressing the time-harmonic dependence, the time independent equation of elasticity

\[
c_s^2 \nabla^2 u(r) + (c_P^2 - c_s^2) \nabla(\nabla \cdot u(r)) + \omega^2 u(r) = 0, \quad r \in V
\]

(10)
where $\omega$ is the eigenfrequency (rads/sec) and 
\[ c_p = \left( \frac{\lambda + 2\mu}{\rho} \right)^{\frac{1}{2}}, \quad c_s = \left( \frac{\mu}{\rho} \right)^{\frac{1}{2}} \]
are the velocities of the longitudinal and transverse waves, respectively, \( \lambda, \mu \) are the Lamè constants and \( \rho \) is the skull density.

The displacement field is supposed to satisfy the boundary conditions

\[ \mathbf{T}\mathbf{u}(\mathbf{r}) = \mathbf{0}, \quad r \in S_1, S_2 \quad (11) \]

where

\[ \mathbf{T} = 2\mu \hat{n} \cdot \nabla + \lambda \hat{n} \nabla \cdot + \mu \hat{n} \times \nabla \times \]

is the stress operator on every surface having \( \hat{n} \) as its exterior unit normal vector.

Following Helmholtz decomposition we lead to the conclusion that the displacements field can be respresented as the superposition of an irrotational longitudinal field \( \mathbf{u}^l(\mathbf{r}) \) and a solenoidal transverse one \( \mathbf{u}^t(\mathbf{r}) \). These fields can be represented through scalar functions satisfying Helmholtz equation. More precisely

\[ \mathbf{u}^l(\mathbf{r}) \in \left\{ \nabla \Phi_p(\mathbf{r}) : (\Delta + k_p^2)\Phi_p(\mathbf{r}) = 0 \right\} \quad (12) \]

\[ \mathbf{u}^t(\mathbf{r}) \in \left\{ \nabla \Phi_p(\mathbf{r}) \times (\mathbf{r} - \alpha \hat{z}) : \nabla \times [\nabla \Phi_p(\mathbf{r}) \times (\mathbf{r} - \alpha \hat{z})] : (\nabla^2 + k_s^2)\Phi_s(\mathbf{r}) = 0 \right\} \quad (13) \]

where \( k_s = \frac{\omega}{c_s}, \quad k_p = \frac{\omega}{c_p} \) are the wave numbers of the transverse and longitudinal waves, respectively.
Consequently, the equation of elasticity leads to the scalar Helmholtz equation through Helmholtz decomposition.

We note that in (12) and (13) any combination of $\mathbf{r}$ and constant vectors can be used instead of $\mathbf{r} - \alpha \mathbf{z}$. This specific choice is based on the property of $\mathbf{r} - \alpha \mathbf{z}$ to be, uniformly for every sphere $j = \text{const}$, a good approximation of the unit exterior normal vector (especially for large $j$).

The Helmholtz equation in bispherical coordinates does not admit R-separability of coordinates. However, when the eccentricity is considered to be small, Helmholtz equation can be solved appropriately.

Equations (12), (13) indicate that the determination of the elastic fields is based on the solution of the scalar Helmholtz equation

$$\nabla^2 u(\mathbf{r}) + k^2 u(\mathbf{r}) = 0,$$

where $u$ is a scalar function.

We express the Laplacian in bispherical coordinates

$$\nabla^2 = \frac{(\cosh j - \cos \theta)^3}{\alpha^2}$$

$$\left\{ \frac{\partial}{\partial j} \left( \frac{1}{\cosh j - \cos \theta} \frac{\partial}{\partial j} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \frac{\sin \theta}{\cosh j - \cos \theta} \frac{\partial}{\partial \theta} \right) + \frac{1}{\cosh j - \cos \theta} \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right\}$$

We assume that equation (14) accepts a solution of the form:

$$u(\mathbf{r}) = \sqrt{\cosh j - \cos \theta} J(j) \Theta(\theta) \Phi(\phi).$$
Inserting (16) in (14) we find that

\[
\frac{J''(j)}{J(j)} + \frac{\Theta''(\vartheta)}{\Theta(\vartheta)} + \cot \vartheta \frac{\Theta'(\vartheta)}{\Theta(\vartheta)} + \frac{1}{\sin^2 \vartheta} \Phi'(\varphi) - \frac{1}{4} \frac{k^2 \alpha^2}{(\cosh j - \cos \vartheta)^2} = 0.
\]  

(17)

Note that the last term in (17) is responsible for lack of R-separability.

We examine now how the term \( \frac{\alpha^2}{(\cosh j - \cos \vartheta)^2} \) is affected by the eccentricity \( \varepsilon \). For the limiting case of small eccentricity (the spheres tend to become concentric) equations (7), (8), (9) give

\[
\alpha = \frac{r_1^2 - r_2^2}{2\varepsilon},
\]

(18)

\[
e^{-h} = \frac{r_1 \varepsilon}{r_1^2 - r_2^2}, \quad e^{-j} = \frac{r_2 \varepsilon}{r_1^2 - r_2^2}.
\]

(19)

For \( j_1 \leq j \leq j_2 \) and \( \cosh j \gg \cos \vartheta \) we can make the approximation

\[
\frac{\alpha^2}{(\cosh j - \cos \vartheta)^2} \equiv \frac{\alpha^2}{\cosh j^2}.
\]

then equation (17) is R-separable and following separation of variables we get

\[
u(r) \equiv \begin{bmatrix} j_n(k2\alpha e^{-j}) \\ j_n'(k2\alpha e^{-j}) \end{bmatrix} \times P^m_n(\cos \vartheta) \times e^{in\varphi}
\]

(20)

where \( j_n, \ y_n \) stand for the spherical Bessel and Neumann functions of order \( n \), respectively and \( P^m_n(\cos \vartheta) \) are the associated Legendre functions.
Simultaneously, the equations (1), (2) and (3), relating cartesian and bispherical coordinates, take the limiting form

\[ x = (2ae^{-j}) \sin \theta \cos \phi \]
\[ y = (2ae^{-j}) \sin \theta \sin \phi \]
\[ z = \alpha + (2ae^{-j}) \cos \theta. \]  

Equations (21) for \( r = 2ae^{-j} \) represent the spherical coordinates and equations (20) constitute the solution of the Helmholtz equation in spherical coordinates.

Nevertheless, the assumption \( \frac{\alpha}{\cosh j - \cos \theta} = \frac{\alpha}{\cosh j} \) corresponds to the case of concentric spheres with common center \( \alpha \coth j_1 = \alpha \coth j_2 = \alpha \) as it is induced by (21) for \( j = j_1 \) and \( j = j_2 \), and so this assumption degenerates the problem to the spherical hollow system.

However, for the typical values of eccentricity occurring in the human skull system (for skull nonuniformities and the presence of facial bones) the expansion of \( \frac{\alpha^2}{(\cosh j - \cos \theta)^2} \) in terms of powers of \( \frac{\cos \theta}{\cos j} \) and the restriction to the first two terms, instead of the first alone, is a very good approximation.

Adopting the approximation

\[ \frac{\alpha^2}{(\cosh j - \cos \theta)^2} \equiv \frac{\alpha^2}{\cosh j^2} \left(1 + 2\frac{\cos \theta}{\cosh j}\right) \equiv (2ae^{-j})^2 \left(1 + 4\frac{\cos \theta}{e^j}\right). \]  

equation (14) becomes

\[ 10 \]
\[ \frac{\partial^2 \ddot{u}}{\partial j^2} + \frac{\partial^2 \ddot{u}}{\partial \theta^2} + \cot \theta \frac{\partial \ddot{u}}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2 \ddot{u}}{\partial \phi^2} - \frac{1}{4} \ddot{u} + \frac{4k^2 \alpha^2}{e^{2j}} \ddot{u} + \frac{16k^2 \alpha^2}{e^{4j}} \cos \theta \ddot{u} = 0, \quad (23) \]

where

\[ u(j, \theta, \phi) = \sqrt{\cosh j - \cos \theta} \ddot{u}(j, \theta, \phi) \quad (24) \]

which does not \text{R}-separate coordinates, but it accepts a specific treatment. It is known that in the case of concentric spheres the eigenfrequency spectrum does not depend on the azimuthal coordinate \( \phi \) [6]. Similarly, because our main interest is the examination of the shifting of the eigenfrequency spectrum from the one of the concentric spheres case, we are looking for solutions independent of \( \phi \).

In addition, following Lagrange method arguments and based on completeness of Legendre functions in the space of square - integrable functions in the interval [-1,1] we represent the solution of (23) in the form

\[ \ddot{u} = \sum_{n} \hat{G}_n(j) P_n(\cos \theta), \quad (25) \]

where summation extends, a priori, over \( \mathbb{N}^* \).

Replacing equation (25) in (23), using the fact that \( P_n(\cos \theta) \) satisfy Legendre equation and the relation

\[ \cos \theta P_n(\cos \theta) = \frac{n}{2n+1} P_{n-1}(\cos \theta) + \frac{n+1}{2n+1} P_{n+1}(\cos \theta), \quad (26) \]
we get that

$$\sum_{n} (Z_n \hat{G}_n) P_n + \frac{16k^2 \alpha^2}{e^{3j}} \sum_{n} \left[ \frac{n}{2n+1} P_{n-1} \hat{G}_n + \frac{n+1}{2n+1} P_{n+1} \hat{G}_n \right] = 0, \quad (27)$$

where the arguments of the participating functions for simplicity have been omitted and $Z_n$ is the operator

$$Z_n = \frac{\partial^2}{\partial j^2} - \left( n + \frac{1}{2} \right)^2 + \frac{4k^2 \alpha^2}{e^{3j}}. \quad (28)$$

Taking into account the orthogonality of Legendre functions $P_n(\cos \vartheta)$ we conclude that for each $n$ there exists, induced by (27), the following solution of equation (23),

$$\hat{h}^{(n)} = \hat{G}_{n-1}^{(n)}(j) P_{n-1}(\cos \vartheta) + \hat{G}_n^{(n)}(j) P_n(\cos \vartheta) + \hat{G}_{n+1}^{(n)}(j) P_{n+1}(\cos \vartheta) \quad (29)$$

which mixes the central and the two consecutive Legendre functions, where the functions $\hat{G}_{n-1}^{(n)}$, $\hat{G}_n^{(n)}$, $\hat{G}_{n+1}^{(n)}$ satisfy the following equations

$$Z_n \hat{G}_n^{(n)} = -\frac{16k^2 \alpha^2}{e^{3j}} \left[ \frac{n+1}{2n+3} \hat{G}_{n+1}^{(n)} + \frac{n}{2n-1} \hat{G}_{n-1}^{(n)} \right] \quad (30)$$

$$Z_{n+1} \hat{G}_{n+1}^{(n)} = -\frac{16k^2 \alpha^2}{e^{3j}} \frac{n+1}{2n+1} \hat{G}_n^{(n)} \quad (31)$$

$$Z_{n-1} \hat{G}_{n-1}^{(n)} = -\frac{16k^2 \alpha^2}{e^{3j}} \frac{n}{2n+1} \hat{G}_n^{(n)}. \quad (32)$$

Since the quantity $2\alpha e^{-j}$ constitutes the radial coordinate in the limit $\frac{\cos \vartheta}{e^{3j}} \to 0$, where the spherical geometry appears as limiting case of the bispherical one, it consists of an indicative characteristic dimension of the bispherical shell for $j_1 \leq j \leq j_2$. In addition,
given that \( \frac{\cos \theta}{e^{j}} \leq e^{-j} \) and that the term \( \frac{\alpha}{\cosh j - \cos \theta} \) is expanded in terms of \( \left( \frac{\cos \theta}{e^{j}} \right)^{n}, \ n = 0, 1, 2, \ldots \), the term \( e^{-j} \) is the characteristic quantity specifying the order of the desirable approximation.

In what follows we decided to keep the first two terms of the expansion and ignore terms proportional to \( (e^{-j})^{n} \) with \( n \geq 2 \). We note that considering only the first term in the expansion of \( \frac{\alpha}{\cosh j - \cos \theta} \) we find that

\[
\hat{\bar{u}}(r) \approx \begin{cases} 
J_{\gamma \frac{1}{2}}(k(2\alpha e^{-j})) \\
Y_{\gamma + \frac{1}{2}}(k(2\alpha e^{-j})) 
\end{cases} \times P_{\gamma}(\cos \theta) \quad (33)
\]

in accordance with (20) for \( m = 0 \), where \( J_{\gamma}, Y_{\gamma} \) are the Bessel and Neumann functions, respectively.

For zeroth order approximation we have

\[
\hat{\bar{G}}^{(\gamma),0}_{n+1} = \hat{\bar{G}}^{(\gamma),0}_{n-1} = 0
\]

and

\[
\hat{\bar{G}}^{(\gamma),0}_{n} \in \begin{cases} 
J_{n+\frac{1}{2}}(k(2\alpha e^{-j})) \\
Y_{n+\frac{1}{2}}(k(2\alpha e^{-j})) 
\end{cases} . 
\quad (34)
\]

Thus, in the first order approximation we have
\[
\hat{G}^{(a)}(j) = \hat{G}^{(a),0} + O(e^{-j}) \\
\hat{G}^{(e)}(j) = O(e^{-j}) \\
\hat{G}^{(e)}(j) = O(e^{-j}).
\] (35)

Omitting terms with order greater than one, and taking into account that the operator \( Z_n \) does not alter the order of the function on which acts we can transform the system of equations (30) - (32) as follows

\[
Z_n \hat{G}^{(a)}(j) = 0
\] (36)

\[
Z_{n+1} \hat{G}^{(e)}(j) = -\frac{16k^2\alpha^2}{e^{3j}} \frac{n+1}{2n+1} \hat{G}^{(a),0}
\] (37)

\[
Z_{n+1} \hat{G}^{(e)}(j) = -\frac{16k^2\alpha^2}{e^{3j}} \frac{n}{2n+1} \hat{G}^{(a),0}
\] (38)

Consequently,

\[
\hat{G}^{(a)}(j) = \hat{J}^{(a)}(j) = \begin{cases} 
J_{\frac{n+1}{2}}(k2\alpha e^{-j}), & l = 1 \\
Y_{\frac{n+1}{2}}(k2\alpha e^{-j}), & l = 2
\end{cases}
\] (39)

and

\[
\hat{G}^{(e)}(j) = c_1 \hat{J}^{(e)}_{n+1}(j) + c_2 \hat{J}^{(e)}_{n+1}(j) + \text{partial solution of (37)}
\]
\[
\hat{G}^{(e)}(j) = c_3 \hat{J}^{(e)}_{n+1}(j) + c_4 \hat{J}^{(e)}_{n+1}(j) + \text{partial solution of (38)}
\] (40)

The terms \( c_1 \hat{J}^{(e)}_{n+1}(j) + c_2 \hat{J}^{(e)}_{n+1}(j) \) and \( c_3 \hat{J}^{(e)}_{n+1}(j) + c_4 \hat{J}^{(e)}_{n+1}(j) \) are solutions to the corresponding homogeneous equation to (37) (resp. (38)) and combined with \( P_{n+1}(\cos \theta) \) (resp. \( P_{n+1}(\cos \theta) \)) give terms which can be incorporated to \( \hat{u}^{(n+1)} \) solution (resp. \( \hat{u}^{(n-1)} \)) in \( \hat{G}^{(e+1)} \) (resp. \( \hat{G}^{(e-1)} \)) term.
Thus, in order to find $\hat{G}_{n \alpha}^{(s)}(j)$ we have to determine a partial solution for every one of the equations (37) and (38) for the cases $l = 1, 2$ (actually we have to solve four equations).

We describe the solution procedure of eq. (37) with $l = 1$. The other cases are treated similarly. More precisely we have the differential equation:

$$Z_{v+1, n+1, \hat{G}_{n \alpha}^{(s)}} = -\frac{16 k^2 \alpha^2}{e^{2j}} \frac{n + 1}{2n + 1} \hat{J}^s_{n \alpha}(j).$$  \hspace{1cm} (41)

Following Lagrange method we assume that

$$\hat{G}_{n \alpha}^{(s)}(j) = A(j) J_{n+\frac{1}{2}}(2 \alpha e^{-j}) + B(j) Y_{n+\frac{1}{2}}(2 \alpha e^{-j})$$  \hspace{1cm} (42)

where $A(j), B(j)$ functions to be determined. This leads to the system

$$J_{n+\frac{1}{2}}(2 \alpha e^{-j})A'(j) + Y_{n+\frac{1}{2}}(2 \alpha e^{-j})B'(j) = 0$$

$$\frac{d}{dj} J_{n+\frac{1}{2}}(2 \alpha e^{-j})A'(j) + \frac{d}{dj} Y_{n+\frac{1}{2}}(2 \alpha e^{-j})B'(j) = -\frac{16 k^2 \alpha^2}{e^{2j}} \frac{n + 1}{2n + 1} J_{n+\frac{1}{2}}(2 \alpha e^{-j}).$$

By using that

$$\begin{vmatrix}
J_{n+\frac{1}{2}}(2 \alpha e^{-j}) & Y_{n+\frac{1}{2}}(2 \alpha e^{-j}) \\
\frac{d}{dj} J_{n+\frac{1}{2}}(2 \alpha e^{-j}) & \frac{d}{dj} Y_{n+\frac{1}{2}}(2 \alpha e^{-j})
\end{vmatrix} = \frac{2}{\pi}$$
the solution of the previous system results in the following expression for \( \hat{G}_{n+1,1}^{(n)}(j) \),

\[
\hat{G}_{n+1,1}^{(n)}(j) = -8\pi k^2 \alpha^2 \frac{n + 1}{2n + 1} \begin{bmatrix}
J_{n+1/2}^{-1}(k2\alpha e^{-j}) & J_{n+1/2}^{-1}(k2\alpha e^{-j}) e^{-3j} df \\
-Y_{n+1/2}^{-1}(k2\alpha e^{-j}) & J_{n+1/2}^{-1}(k2\alpha e^{-j}) e^{-3j} df 
\end{bmatrix}
\]

(43)

The integrals are handled by passing to spherical Bessel functions. Indeed, the second integral becomes

\[
\frac{2}{\pi} \int \frac{J_{n+1/2}^{-1}(k2\alpha e^{-j}) J_{n+3/2}^{-1}(k2\alpha e^{-j}) e^{-3j} df} = \]

\[
\frac{1}{4\pi k^2 \alpha^3} \int \frac{J_n(\omega') J_{n+1}(\omega') \omega' d\omega'}
\]

where \( \omega = k2\alpha e^{-j} \).

On the other hand

\[
I = \int \frac{J_n(\omega') J_{n+1}(\omega') \omega' d\omega'}
\]

\[
= -\int \omega'^{2n+4} \left( \frac{1}{\omega'} \frac{d}{d\omega'} \right)^n \left( \frac{\sin \omega'}{\omega'} \right) \left( \frac{1}{\omega'} \frac{d}{d\omega'} \right)^{n+1} \left( \frac{\sin \omega'}{\omega'} \right) d\omega' 
\]

\[
= -\omega^3 J_n^2(\omega) + \omega^{2n+4} \left( \frac{1}{\omega'} \frac{d}{d\omega'} \right)^n \left( \frac{\sin \omega'}{\omega'} \right) \left( \frac{1}{\omega'} \frac{d}{d\omega'} \right)^n \left( \frac{\sin \omega'}{\omega'} \right) d\omega' 
\]

\[
+ \int (2n + 3) \omega^{2n+2} \left[ \left( \frac{1}{\omega'} \frac{d}{d\omega'} \right)^n \left( \frac{\sin \omega'}{\omega'} \right) \right]^2 d\omega'
\]

or

- 16 -
$$2I = -\omega^3 j_n^2(\omega) + (2n+3) \frac{\omega^2}{2} j_n^2(\omega) d\omega' = -\omega^3 j_n^2(\omega) + (2n+3) \frac{\pi^2}{2} j_{\frac{n+1}{2}}(\omega') d\omega'.$$

In addition we have [10]

$$\int_{\frac{n+1}{2}}^{\omega} \omega' J{n+\frac{1}{2}}^2(\omega') d\omega' = \frac{\omega^2}{2} \left\{ J{n+\frac{1}{2}}^2(\omega) - J{\frac{n-\frac{1}{2}}{2}}^2(\omega) J{\frac{n+\frac{1}{2}}{2}}^2(\omega) \right\}. \quad (44)$$

Consequently

$$I = -\frac{1}{2} \omega^3 j_n^2(\omega) + (2n+3) \frac{\omega^3}{4} \left\{ j_n^2(\omega) - j_{n-1}(\omega) j_{n+1}(\omega) \right\}$$

$$= \frac{1}{4} \omega^3 \left[ (2n+1) j_n^2(\omega) - (2n+3) j_{n+1}(\omega) j_{n-1}(\omega) \right].$$

We infer that

$$\int_{\frac{n+1}{2}}^{\omega} (k^2 \omega e^{-j}) J{n+\frac{1}{2}}(k^2 \omega e^{-j}) e^{-j\gamma} df =$$

$$- \frac{1}{4 \pi k^3 \omega^3} \frac{1}{4} \omega^3 \left[ (2n+1) j_n^2(\omega) - (2n+3) j_{n+1}(\omega) j_{n-1}(\omega) \right].$$

Following the same arguments and proving similarly to (44) that

$$\int_{\frac{n+1}{2}}^{\omega} \omega' J_n(\omega') Y_n(\omega') d\omega' = \frac{\omega^2}{4} \left[ 2J_n(\omega) Y_n(\omega) - J_{n-1}(\omega) Y_{n+1}(\omega) - J_{n+1}(\omega) Y_{n-1}(\omega) \right]$$

we find that the first integral of (43) to be
\[ \int \frac{J_{\frac{n+1}{2}}(k2\alpha e^{-j}) Y_{\frac{n+1}{2}}(k2\alpha e^{-j}) e^{-3j} dj = \frac{1}{4\pi \alpha^2} \left\{ \frac{\omega^3}{6} + \frac{\omega^3}{2} j_n(\omega) x_n(\omega) - \frac{2n+3}{2} \int \omega^2 j_n(\omega) x_n(\omega) d\omega \right\}. \]

The equation (43) takes the form

\[ \hat{G}^{(n)}_{\kappa+1,1} = -\sqrt{\frac{2}{\pi}} \frac{1}{n+1} j_{n+1}(\omega) e^{-j} \left[ (2n+1)\omega^2 j_n(\omega) - (2n+3)\omega^2 j_{n+1}(\omega) j_{n-1}(\omega) \right] \]
\[ -\sqrt{\frac{2}{\pi}} \frac{1}{2n+1} j_{n-1}(\omega) e^{-j} \left[ \frac{\omega^2}{6} + \frac{\omega^2}{2} j_n(\omega) x_n(\omega) \right] \]
\[ \left[ \frac{2n+3}{8} \left[ 2\omega^2 j_n(\omega) x_n(\omega) - \omega^2 j_{n-1}(\omega) x_{n-1}(\omega) - \omega^2 j_{n+1}(\omega) x_{n+1}(\omega) \right] \right] \]

Similarly, we can define the functions \( \hat{G}^{(n)}_{\kappa+1,1}, \hat{G}^{(n)}_{\kappa-1,1}, \hat{G}^{(n)}_{\kappa+1,2} \).

Consequently, we lead to the solutions

\[ u^{(n),l} = \sqrt{\cosh j - \cos \theta} \sum_{i=1}^{n+1} \hat{G}^{(n)}_{\kappa+i,1}(j) P_{\kappa+i}(\cos \theta). \]  \( (45) \)

Omitting terms of order greater than one in the expansion of \( \frac{\cos \theta}{\cosh j} \), we finally obtain

\[ u^{(n),l} = \left[ G^{(n)}_{\kappa+1,1} - \frac{n}{2n+1} e^{-j} G^{(n)}_{\kappa+1,1} \right] P_{\kappa+1} + G^{(n)}_{\kappa,1} P_{\kappa} + \left[ G^{(n)}_{\kappa-1,1} - \frac{n+1}{2n+1} e^{-j} G^{(n)}_{\kappa-1,1} \right] P_{\kappa-1} \]  \( (46) \)

with

\[ G^{(n)}_{\kappa,l} \begin{cases} j_n(\omega), & l = 1 \\ y_n(\omega), & l = 2 \end{cases} \]  \( (47) \)
\[
G_{n+1}^{(n)}(\omega) = -\frac{n}{2n+1} \frac{1}{\omega} \left[ (2n+1)\omega^2 j_n^2(\omega) - (2n+3)\omega^2 j_{n+1}(\omega)j_{n-1}(\omega) \right]
\]

\[
-4 j_{n+1}(\omega) \frac{n+1}{2n+1} e^{-i\omega t} \left\{ \frac{\omega^2}{6} + \frac{\omega^2}{2} j_n(\omega) y_n(\omega) - \frac{2n+3}{8} \omega^2 [2 j_n(\omega) y_n(\omega) - j_{n+1}(\omega) y_{n+1}(\omega) - j_{n-1}(\omega) y_{n-1}(\omega)] \right\}
\]

\[
G_{n+1}^{(n)}(\omega) = j_{n+1}(\omega) \frac{n+1}{2n+1} e^{-i\omega t} \left\{ \frac{\omega^2}{6} + \frac{\omega^2}{2} j_n(\omega) y_n(\omega) - \frac{2n+3}{8} \omega^2 [2 j_n(\omega) y_n(\omega) - j_{n+1}(\omega) y_{n+1}(\omega) - j_{n-1}(\omega) y_{n-1}(\omega)] \right\}
\]

\[
G_{n+1}^{(n)}(\omega) = -\frac{n}{2n+1} \frac{1}{\omega} \left[ (2n+1)\omega^2 j_n^2(\omega) - (2n+3)\omega^2 j_{n+1}(\omega)j_{n-1}(\omega) \right]
\]

\[
-4 j_{n-1}(\omega) \frac{n}{2n+1} e^{-i\omega t} \left\{ \frac{\omega^2}{6} + \frac{\omega^2}{2} j_{n-1}(\omega) y_{n-1}(\omega) - \frac{2n+1}{8} \omega^2 [2 j_{n-1}(\omega) y_{n-1}(\omega) - j_{n-2}(\omega) y_{n-2}(\omega) - j_n(\omega) y_n(\omega)] \right\}
\]

\[
G_{n+1}^{(n)}(\omega) = j_{n+1}(\omega) \frac{n}{2n+1} e^{-i\omega t} \left\{ \frac{\omega^2}{6} + \frac{\omega^2}{2} j_{n-1}(\omega) y_{n-1}(\omega) - \frac{2n+1}{8} \omega^2 [2 j_{n-1}(\omega) y_{n-1}(\omega) - j_{n-2}(\omega) y_{n-2}(\omega) - j_n(\omega) y_n(\omega)] \right\}
\]

After the determination of the solutions \(u^{(n,i)}\) of the Helmholtz equation, we construct the Navier eigenfunctions

\[
L_\alpha'(r') = \nabla u^{(n,i)}(r';k_p)
\]

\[
M_\alpha'(r') = \nabla \times u^{(n,i)}(r';k_p) \times (r - \alpha \hat{z})
\]

\[
N_\alpha'(r') = \nabla \times M_\alpha'(r')
\]

The operator \(\nabla\) in bispherical coordinates has the form
\[ \nabla = \frac{1}{\alpha} (\cosh j - \cos \vartheta) \left( j \frac{\partial}{\partial j} + \vartheta \frac{\partial}{\partial \vartheta} \right) \]  \hspace{1cm} (53)

and must be applied appropriately (as Eq. (52) defines) on the function \( u^{(n)}(r') \) given by (46) to produce Navier eigenfunctions. The analytical procedure is very complex. However, the determination of the Navier eigenfunctions is an intermediate step in order to satisfy the boundary conditions.

Indeed, the completeness of Navier eigenfunctions allows for the following representation of the elastic field

\[ u(r') = \sum_{n=0}^{\infty} \sum_{l=1}^{2} \left[ \alpha_n L_n^l (r') + \beta_n M_n^l (r') + \gamma_n N_n^l (r') \right]. \hspace{1cm} (54) \]

The boundary conditions (11) impose that

\[ T u(r') = \sum_{n=0}^{\infty} \sum_{l=1}^{2} \left[ \alpha_n^{l'} T L_n^l (r') + \beta_n^{l'} T M_n^l (r') + \gamma_n^{l'} T N_n^l (r') \right] = 0, \hspace{0.5cm} r' \in S_1, S_2. \hspace{1cm} (55) \]

Therefore, we have to determine the functions \( TL_n^l (r') \), \( TM_n^l (r') \) and \( TN_n^l (r') \).

Taking the normal unit vector \( \hat{n} \) on the surfaces \( S_1, S_2 \) to be \(-\hat{j}\) we find easily that

\[ TL_n^l (r') = -\left[ 2 \mu \hat{j} \cdot \nabla L_n^l (r') - \lambda k_n^2 \hat{j} u_n^{(n)}(r') \right] \hspace{1cm} (56) \]
\[ TM_n^l (r') = -\left[ 2 \mu \hat{j} \cdot \nabla M_n^l (r') + \mu \hat{j} \times N_n^l (r') \right] \hspace{1cm} (57) \]
\[ TN_n^l (r') = -\left[ 2 \mu \hat{j} \cdot \nabla N_n^l (r') + \mu k_n^2 \hat{j} \times M_n^l (r') \right]. \hspace{1cm} (58) \]
We define the functions

\[
A_{n+1,l}^{(s)}(j;k_l) = G_{n+1,l}^{(s)}(j;k_l) - \frac{n}{2n+1} e^{-j} G_{n,l}^{(s)}(j;k_l)
\]

\[
A_{n,l}^{(s)}(j;k_l) = G_{n,l}^{(s)}(j;k_l)
\]

\[
A_{n+1,l}^{(s)}(j;k_l) = G_{n+1,l}^{(s)}(j;k_l) - \frac{n+1}{2n+1} e^{-j} G_{n,l}^{(s)}(j;k_l)
\]

where \( k_l, \ i = p,s \) indicate the longitudinal and transverse wave numbers, respectively.

For simplicity in what follows we denote \( A_{n+1,l}^{(s),p} = A_{n+1,l}^{(s)}(j;k_p) \) and \( A_{n+1,l}^{(s),p} = A_{n+1,l}^{(s)}(j;k_s) \) for \( i = -1,0,1; \ l = 1,2 \).

Finally, we lead to the following expressions

\[
\text{TL}_l = -2\mu_j \left\{ \begin{array}{c}
-k_f^2 \sum_{l=1}^{1} A_{n+1,l}^{(s),p} P_{n+1}(\cos \vartheta) - \frac{4}{r^2} \cos \vartheta e^{-j} \left[ \frac{\partial A_{n+1,l}^{(s),p}}{\partial j} + A_{n+1,l}^{(s),p}(n+1) \right] P_n(\cos \vartheta) \\
+ \frac{1}{r^2} \sum_{l=1}^{1} \left[ 2 \frac{\partial}{\partial j} A_{n+1,l}^{(s),p} + (n+i)(n+i+1) A_{n+1,l}^{(s),p} \right] P_{s+1}(\cos \vartheta) \\
+ \frac{2}{r^2} \sin \vartheta e^{-j} \left[ 2 \frac{\partial}{\partial j} A_{n+1,l}^{(s),p} + A_{n+1,l}^{(s),p} \right] \frac{\partial}{\partial \vartheta} P_n(\cos \vartheta) \\
- \frac{2}{r^2} \cos \vartheta e^{-j} \left[ 2 \frac{\partial}{\partial j} A_{n+1,l}^{(s),p} + A_{n+1,l}^{(s),p} \right] \frac{\partial}{\partial \vartheta} P_{s+1}(\cos \vartheta) \\
\end{array} \right\}
\]

\[
+ \lambda k_f^2 \sum_{l=1}^{1} A_{n+1,l}^{(s),p} P_{n+1}(\cos \vartheta),
\]

(60)
\[ \mathbf{TM}_s = -2\mu \hat{\varphi} \left\{ -\frac{1}{r} \sum_{i=1}^{i=\infty} \frac{\partial}{\partial \varphi} A_{n+\ell_i}^{(s),s} \frac{\partial}{\partial \vartheta} P_{n+1}(\cos \vartheta) \right. \\
+ \frac{3\cos \vartheta}{r} e^{-\varphi} \frac{\partial}{\partial \varphi} A_{n+\ell_i}^{(s),s} \frac{\partial}{\partial \vartheta} P_n(\cos \vartheta) - \frac{1}{r} \frac{\partial}{\partial \vartheta} \frac{\partial}{\partial \vartheta} P_n(\cos \vartheta) \\
+ \left. \left[ n(n+1) \right] \frac{\partial}{\partial \vartheta} \frac{\partial}{\partial \vartheta} P_n(\cos \vartheta) - \frac{1}{r} \frac{\partial}{\partial \vartheta} \frac{\partial}{\partial \vartheta} P_n(\cos \vartheta) \right\} \\
+ 2\mu \hat{\varphi} \left\{ -\frac{1}{r} \sum_{i=1}^{i=\infty} A_{n+\ell_i}^{(s),s} \frac{\partial}{\partial \vartheta} P_{n+1}(\cos \vartheta) - 2\cos \vartheta e^{-\varphi} A_{n+\ell_i}^{(s),s} \frac{\partial}{\partial \vartheta} P_n(\cos \vartheta) \right\} \\
+ \mu k_j^2 A_{n+\ell_i}^{(s),s} P_n(\cos \vartheta) r \sin e^{-\varphi} \hat{\varphi}, \right. \\
\] 

(61)
\[ +4 \frac{\mu}{r^3} e^{-i\phi} \hat{\partial} A_{n,n}^{(n)} \left[ \cos \phi \frac{\partial}{\partial \phi} P_n(\cos \phi) + n(n+1) \sin \phi P_n(\cos \phi) \right] \]
\[ -4 \frac{\mu}{r^3} e^{-i\phi} j \hat{\partial} A_{n,n}^{(n)} n(n+1) \cos \phi P_n(\cos \phi) + 4 \frac{\mu}{r^3} e^{-i\phi} \hat{\partial} A_{n,n}^{(n)} \cos \phi \frac{\partial}{\partial \phi} P_n(\cos \phi). \] (62)

Inserting (60), (61), (62) in (55) and exploiting the orthogonality of Legendre functions, we lead to the following linear algebraic system \(((6 \times (n'+1)) \times (6 \times (n'+1)))\), where \(n'\) indicates the truncation level

\[
\begin{bmatrix}
D_0^{(0)} & D_1^{(1)} & 0 & 0 & 0 & \cdots & x_0 \\
D_0^{(1)} & D_1^{(2)} & D_2^{(1)} & 0 & 0 & \cdots & x_1 \\
0 & D_1^{(2)} & D_2^{(2)} & D_3^{(1)} & 0 & \cdots & x_2 \\
0 & 0 & D_2^{(3)} & D_3^{(2)} & D_4^{(1)} & \cdots & x_3 \\
0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix} \begin{bmatrix}
x_0 \\
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix} = 0
\] (63)

where
\[ x_i = [\alpha_i, \alpha_i, \beta_i, \beta_i, \gamma_i, \gamma_i]^T \]

and the matrices \(D_n^{(n)}, D_{n+1}^{(n)}, D_{n+1}^{(n+1)}\) are \(6 \times 6\) and their elements are given in Appendices A, B, C, respectively.

The system of equations (63) can be written in compact form as
\[
Dx = 0. \quad \text{(64)}
\]

In order the system (64) to have nontrivial solutions we must have
\[
\det(D) = 0. \quad \text{(65)}
\]

Condition (65) provides the characteristic (frequency) equation, the roots of which are the eigenfrequencies \(\omega_i\) of the system under discussion.
4. Numerical Results and Discussion

The frequency equation (65) has been solved numerically by using a matrix determinant computation routine along with a bisection method. The dimension of the matrix $D$ depends on the value of $n'$, which ensures convergence of $\omega_k$, $k = 1, 2, ..., 20$. The computation of $\omega_k = \omega_k(\varepsilon, n')$ has been done by using an iterative procedure and the results obtained are cited in Table 1. We observe that the value of $n'$ for convergence is strongly dependent on the values of $\varepsilon$, $n'_{conv.} = n'(\varepsilon)$. In the case under discussion, for fixed values of $\varepsilon$, the computational procedure was repeated until $\|\omega_k^{(x+1)} - \omega_k^{(x+1)}\|_1 = O(10^{-4})$, $k = 1, 2, ..., 20$. It is noted that the accuracy of the bisection method used is of the order of $O(10^{-8})$ and the computing time needed is an increasing function of $n'$.

The elements of $D$ are functions of $\omega$ (see Appendices A, B and C) and therefore they have to be computed for every different value of it. This fact requires the computation of Spherical Bessel functions of the first and second kind as well as their derivatives. In our computations we have used Seed's method [11] for the computation of Bessel functions of fractional order and their derivatives by using the definitions of spherical Bessel functions for integer $n'$. We note that recursive relations, although they offer flexibility and fast computations, are not helpful for high values of $n'$ and values of the argument close to zero.

The numerical computations for the system under discussion have been performed by using the following properties for the dry skull:

$$E = 6.5 \times 10^9 \, N \, / \, m^2, \quad \nu = 0.25, \quad \rho = 2.1326 \times 10^5 \, Kg \, / \, m^3,$$
where $E$ and $\nu$ denote the Young's modulus and Poisson's ratio, respectively.

The geometry of the base case (two concentric spheres, Ref. 6) is defined by:

$$r_1 = 0.082 \, m, \quad r_2 = 0.076 \, m.$$

Taking into account the morphology of the human skull we can consider eccentricities $\varepsilon$ varying as

$$\varepsilon \in [0.0, 0.004 \, m].$$

In our computations the determination of the eccentricity is given by a new parameter $\delta$ which gives the eccentricity as a percentage of the inner sphere radius, i.e.

$$\varepsilon = \delta r_2.$$

The effect of the eccentricity on the frequency spectrum of the human dry skull is given in Table 2. We observe that the deviation from the base case introduces a new pattern of eigenfrequencies and shift of the existing eigenfrequencies. The new eigenfrequencies are close to zero but for larger values of $\delta$ those eigenfrequencies enter the frequency spectrum (see Fig 2a and Fig2b).

The variation of $\omega_k$ with inner sphere radius $r_2$ is shown in Table 3. We observe that the inner radius (or the thickness of the skull) is a vital parameter in the determination of the eigenfrequency spectrum and its deviation from the base case. Table 5 gives the frequency spectrum of the system for Poisson's ratio $\nu = 0.2, \nu = 0.25$ and $\nu = 0.3$. As the Poisson's ratio becomes smaller the effect of the eccentricity on the shifting of the eigenfrequency spectrum becomes more significant and the new eigenfrequencies entering the spectrum are larger.
From the results obtained we lead to the following:

- the eccentricity of the human skull plays an important role on the dynamic characteristics of the human head

- facial bones and thickness nonuniformities should be considered in detailed models of the system

Such a detailed morphology model for the human head is under preparation and it will appear in a future communication.
Table 1: Convergence of $\omega_k$, $k = 1, 2, ..., 22$. 
($E = 6.5 \times 10^9 N / m^2$, $\nu = 0.25$, $\rho = 2.1326 \times 10^3 Kg / m^3$, $r_1 = 0.082 m$, $r_2 = 0.076 m$, $r = 0.01r_2$)

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Table 2: Variation of $\omega_1$, $k = 1, 2, ..., 18$ with $\varepsilon = \delta r_2$, $\delta \in [0, 0.01]$.

($E = 6.5 \times 10^4 N/m^2$, $\nu = 0.25$, $\rho = 2.1326 \times 10^3 Kg/m^3$, $r_1 = 0.082m$, $r_2 = 0.076m$.)

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Figure 2a: Variation of $\omega_k$, $k = 9, \ldots, 18$ with $\varepsilon = \delta r_2$, $\delta \in [0.001]$. 
$E = 6.5 \times 10^9 N/m^2$, $\nu = 0.25$, $\rho = 2.1326 \times 10^3 Kg/m^3$, $r_1 = 0.082m$, $r_2 = 0.076m$. 

![Graph showing variations of $\omega_k$ with $\delta$](image-url)
Figure 2b: Variation of $\omega_k, k = 1, \ldots, 4$ with $\varepsilon = \delta r_2$, $\delta \in [0, 0.01]$.

$E = 6.5 \times 10^9 \, N/m^2$, $\nu = 0.25$, $\rho = 2.1326 \times 10^3 \, Kg/m^3$, $r_1 = 0.082 m$, $r_2 = 0.076 m$. 
### Table 3

Variation of $\omega_{k}$, $k = 1, 2, ..., 16$ with $r_{2} \in [0.041m, 0.076m]$ for $\varepsilon = 0$ and $\varepsilon = 7.6 \times 10^{-4}$.

($E = 6.5 \times 10^{7} N/m^{2}$, $\nu = 0.25$, $\rho = 2.1326 \times 10^{3} Kg/m^{3}$, $r_{1} = 0.082m$)

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Table 4: Variation of $\omega_k$, $k=1,2,...,15$ for different Poisson's ratios 
($\delta = 0.00$, $\delta = 0.01$).
($E = 6.5 \times 10^3 \text{ N/m}^2$, $\rho = 2.1326 \times 10^3 \text{ Kg/m}^3$, $r_1 = 0.082m$, $r_2 = 0.076m$.)

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Acknowledgement

The present work forms part of the project "New Systems for Early Medical Diagnosis and Biotechnological Applications" which is supported by the Greek General Secretariat for Research and Technology through the EU funded R&D Program EPET II.
References


APPENDIX A:

Elements of the diagonal block matrices $D^{(n)}_n$:

\[
d(1,1) = -k_p^2 A_{n,1}^{(n),p}(j_2) + \frac{2}{r_2^2} \frac{\partial}{\partial j} A_{n,1}^{(n),p}(j_2) + \frac{n(n+1)}{r_2^2} A_{n,1}^{(n),p}(j_2) - \frac{\lambda k_p^2}{2} A_{n,1}^{(n),p}(j_2),
\]

\[
d(1,2) = -k_p^2 A_{n,2}^{(n),p}(j_2) + \frac{2}{r_2^2} \frac{\partial}{\partial j} A_{n,2}^{(n),p}(j_2) + \frac{n(n+1)}{r_2^2} A_{n,2}^{(n),p}(j_2) - \frac{\lambda k_p^2}{2} A_{n,2}^{(n),p}(j_2),
\]

\[
d(1,3) = 0, \quad d(1,4) = 0,
\]

\[
d(1,5) = \left( -\frac{1}{r_2^2} \frac{\partial}{\partial j} A_{n,1}^{(n),s}(j_2) - \frac{1}{r_2^2} A_{n,1}^{(n),s}(j_2) \right) n(n+1),
\]

\[
d(1,6) = \left( -\frac{1}{r_2^2} \frac{\partial}{\partial j} A_{n,2}^{(n),s}(j_2) - \frac{1}{r_2^2} A_{n,2}^{(n),s}(j_2) \right) n(n+1),
\]

\[
d(2,1) = -k_p^2 A_{n,1}^{(n),p}(j_1) + \frac{2}{r_1^2} \frac{\partial}{\partial j} A_{n,1}^{(n),p}(j_1) + \frac{n(n+1)}{r_1^2} A_{n,1}^{(n),p}(j_1) - \frac{\lambda k_p^2}{2} A_{n,1}^{(n),p}(j_1),
\]

\[
d(2,2) = -k_p^2 A_{n,2}^{(n),p}(j_1) + \frac{2}{r_1^2} \frac{\partial}{\partial j} A_{n,2}^{(n),p}(j_1) + \frac{n(n+1)}{r_1^2} A_{n,2}^{(n),p}(j_1) - \frac{\lambda k_p^2}{2} A_{n,2}^{(n),p}(j_1),
\]

\[
d(2,3) = 0, \quad d(2,4) = 0,
\]

\[
d(2,5) = \left( -\frac{1}{r_1^2} \frac{\partial}{\partial j} A_{n,1}^{(n),s}(j_1) - \frac{1}{r_1^2} A_{n,1}^{(n),s}(j_1) \right) n(n+1),
\]

\[
d(2,6) = \left( -\frac{1}{r_1^2} \frac{\partial}{\partial j} A_{n,2}^{(n),s}(j_1) - \frac{1}{r_1^2} A_{n,2}^{(n),s}(j_1) \right) n(n+1),
\]

\[
d(3,1) = 0, \quad d(3,2) = 0,
\]

\[
d(3,3) = \frac{1}{r_2} \frac{\partial}{\partial j} A_{n,1}^{(n),s}(j_2) + \frac{1}{r_2} A_{n,1}^{(n),s}(j_2), \quad d(3,4) = \frac{1}{r_2} \frac{\partial}{\partial j} A_{n,1}^{(n),s}(j_2) + \frac{1}{r_2} A_{n,2}^{(n),s}(j_2),
\]

\[
d(3,5) = 0, \quad d(3,6) = 0,
\]

\[
d(4,1) = 0, \quad d(4,2) = 0,
\]

\[
d(4,3) = \frac{1}{r_1} \frac{\partial}{\partial j} A_{n,1}^{(n),s}(j_1) + \frac{1}{r_1} A_{n,1}^{(n),s}(j_1), \quad d(4,4) = \frac{1}{r_1} \frac{\partial}{\partial j} A_{n,1}^{(n),s}(j_1) + \frac{1}{r_1} A_{n,2}^{(n),s}(j_1),
\]

\[
d(4,5) = 0, \quad d(4,6) = 0,
\]
\[ d_{5(1)} = -\frac{1}{r_2} \left( \frac{\partial}{\partial j} A_{n,1}^{(s),p}(j_1) + A_{n,1}^{(s),p}(j_2) \right), \quad d_{5(2)} = -\frac{1}{r_2} \left( \frac{\partial}{\partial j} A_{n,2}^{(s),p}(j_1) + A_{n,2}^{(s),p}(j_2) \right), \]

\[ d_{5(3)} = 0, \quad d_{5(4)} = 0, \]

\[ d_{5(5)} = \frac{k_2^2}{r_2^2} A_{n,1}^{(s),j}(j_2) + \frac{k_2^2}{2} A_{n,2}^{(s),j}(j_2), \quad d_{5(6)} = \frac{k_2^2}{r_2^2} A_{n,1}^{(s),j}(j_2) + \frac{k_2^2}{2} A_{n,2}^{(s),j}(j_2), \]

\[ d_{6(1)} = -\frac{1}{r_2} \left( \frac{\partial}{\partial j} A_{n,2}^{(s),p}(j_1) + A_{n,1}^{(s),p}(j_2) \right), \quad d_{6(2)} = -\frac{1}{r_2} \left( \frac{\partial}{\partial j} A_{n,1}^{(s),p}(j_1) + A_{n,2}^{(s),p}(j_2) \right), \]

\[ d_{6(3)} = 0, \quad d_{6(4)} = 0, \]

\[ d_{6(5)} = \frac{1}{r_2^2} \left( \frac{\partial^2}{\partial j^2} A_{n,1}^{(s),j}(j_1) \right) + \frac{k_2^2}{2} A_{n,1}^{(s),j}(j_1), \quad d_{6(6)} = \frac{1}{r_2^2} \left( \frac{\partial^2}{\partial j^2} A_{n,2}^{(s),j}(j_1) \right) + \frac{k_2^2}{2} A_{n,2}^{(s),j}(j_1). \]

**APPENDIX B:**

Elements of the lower diagonal block matrices \( D_{n-1}^{(s)}: \)

\[ d_{1(1)} = -k_2^2 A_{n,1}^{(s-1),p}(j_2) - \frac{4}{r_2} e^{-j_l} \left( \frac{\partial}{\partial j} A_{n-1,1}^{(s-1),p}(j_2) + n(n-1) A_{n-1,1}^{(s-1),p}(j_2) \right) \frac{n}{2n-1}, \]

\[ + \frac{1}{r_2^2} \left[ 2 \frac{\partial}{\partial j} A_{n,1}^{(s-1),p}(j_2) + n(n+1) A_{n,1}^{(s-1),p}(j_2) \right] \frac{\lambda k_2^2}{2} A_{n,1}^{(s-1),p}(j_2) \]

\[ d_{1(2)} = -k_2^2 A_{n,2}^{(s-1),p}(j_2) - \frac{4}{r_2} e^{-j_l} \left( \frac{\partial}{\partial j} A_{n-1,2}^{(s-1),p}(j_2) + n(n-1) A_{n-1,2}^{(s-1),p}(j_2) \right) \frac{n}{2n-1}, \]

\[ + \frac{1}{r_2^2} \left[ 2 \frac{\partial}{\partial j} A_{n,2}^{(s-1),p}(j_2) + n(n+1) A_{n,2}^{(s-1),p}(j_2) \right] \frac{\lambda k_2^2}{2} A_{n,2}^{(s-1),p}(j_2) \]

\[ d_{1(3)} = 0, \quad d_{1(4)} = 0, \]

\[ d_{1(5)} = -\frac{1}{r_2^2} \frac{\partial}{\partial j} A_{n-1,1}^{(s-1),j}(j_2) n(n+1) + \frac{1}{r_2^2} \frac{\partial}{\partial j} \left( e^{-j_l} A_{n-1,1}^{(s-1),j}(j_2) \right) \frac{n(n-1)}{2n-1}, \]

\[ \frac{1}{r_2} \frac{\partial}{\partial j} \left( e^{-j_l} A_{n-1,1}^{(s-1),j}(j_2) \right) \frac{n}{2n-1} - \frac{1}{r_2} \frac{\partial}{\partial j} \left( e^{-j_l} \frac{\partial}{\partial j} A_{n-1,1}^{(s-1),j}(j_2) \right) \frac{n(n-1)}{2n-1}, \]

\[ - \frac{2}{r_2^2} \frac{\partial}{\partial j} \left( e^{-j_l} A_{n-1,1}^{(s-1),j}(j_2) \right) \frac{n}{2n-1} - \frac{1}{r_2} A_{n,1}^{(s-1),j}(j_2) n(n+1) + \frac{1}{r_2} e^{-j_l} A_{n-1,1}^{(s-1),j}(j_2) \frac{n(n-1)}{2n-1}, \]

\[ + \frac{1}{r_2^2} e^{-j_l} A_{n-1,1}^{(s-1),j}(j_2) \frac{n^2(n-1)}{2n-1} - \frac{2}{r_2^2} e^{-j_l} \frac{\partial}{\partial j} A_{n-1,1}^{(s-1),j}(j_2) \frac{n}{2n-1} - \frac{1}{r_2} e^{-j_l} \frac{\partial}{\partial j} A_{n-1,1}^{(s-1),j}(j_2) \frac{n(n-1)}{2n-1}, \]

\[ + \frac{2}{r_2^2} e^{-j_l} A_{n-1,1}^{(s-1),j}(j_2) \frac{n^2(n-1)}{2n-1} + \frac{4}{r_2} e^{-j_l} \frac{\partial}{\partial j} A_{n-1,1}^{(s-1),j}(j_2) \frac{n^2(n-1)}{2n-1}. \]
\[ d(1,6) = -1 \frac{\partial}{r_1^2 \partial j} A^{(n-1,1), (j_2) n(n+1)} + \frac{1}{r_1^2} \frac{\partial}{\partial j} \left( e^{-ij} A^{(n-1,1), (j_2)} \right) \frac{n(n-1)}{2n-1} \]

\[ \frac{1}{r_2^2} \frac{\partial}{\partial j} \left( e^{-ij} A^{(n-1,1), (j_2)} \right) \frac{n(n-1)}{2n-1} - \frac{1}{r_2^2} \frac{\partial}{\partial j} \left( e^{-ij} \frac{\partial}{\partial j} A^{(n-1,1), (j_2)} \right) \frac{n(n-1)}{2n-1} \]

\[ \frac{2}{r_2^2} \frac{\partial}{\partial j} \left( e^{-ij} A^{(n-1,1), (j_2)} \right) \frac{n}{2n-1} - \frac{1}{r_2^2} A^{(n-1,1), (j_2)} \frac{n}{2n-1} + \frac{1}{r_2^2} \frac{\partial}{\partial j} A^{(n-1,1), (j_2)} \frac{n(n-1)}{2n-1} \]

\[ + \frac{1}{r_2^2} e^{-ij} A^{(n-1,1), (j_2)} \frac{n^2(n-1)}{2n-1} + \frac{4}{r_2^2} e^{-ij} \frac{\partial}{\partial j} A^{(n-1,1), (j_2)} \frac{n^2(n-1)}{2n-1} , \]

\[ d(2,1) = -k_A^{(n-1,1), p, (j)} \frac{4}{r_2^2} e^{-ij} \left( \frac{\partial}{\partial j} A^{(n-1,1), p, (j_2)} + n(n-1) A^{(n-1,1), p, (j_2)} \right) \frac{n}{2n-1} \]

\[ + \frac{1}{r_1^2} \left[ \frac{2}{r_1^2} \frac{\partial}{\partial j} A^{(n-1,1), p, (j_2)} + n(n+1) A^{(n-1,1), p, (j_2)} \right] - \frac{\lambda k_A^{(n-1,1), p, (j_2)}}{2} A^{(n-1,1), p, (j_2)} , \]

\[ d(2,2) = -k_A^{(n-1,1), p, (j)} \frac{4}{r_2^2} e^{-ij} \left( \frac{\partial}{\partial j} A^{(n-1,1), p, (j_2)} + n(n-1) A^{(n-1,1), p, (j_2)} \right) \frac{n}{2n-1} \]

\[ + \frac{1}{r_1^2} \left[ \frac{2}{r_1^2} \frac{\partial}{\partial j} A^{(n-1,1), p, (j_2)} + n(n+1) A^{(n-1,1), p, (j_2)} \right] - \frac{\lambda k_A^{(n-1,1), p, (j_2)}}{2} A^{(n-1,1), p, (j_2)} , \]

\[ d(2,3) = 0 , \quad d(2,4) = 0 , \]

\[ d(2,5) = -1 \frac{\partial}{r_1^2 \partial j} A^{(n-1,1), (j_2) n(n+1)} + \frac{1}{r_1^2} \frac{\partial}{\partial j} \left( e^{-ij} A^{(n-1,1), (j_2)} \right) \frac{n(n-1)}{2n-1} \]

\[ \frac{2}{r_1^2} \frac{\partial}{\partial j} \left( e^{-ij} A^{(n-1,1), (j_2)} \right) \frac{n}{2n-1} - \frac{1}{r_1^2} \frac{\partial}{\partial j} \left( e^{-ij} \frac{\partial}{\partial j} A^{(n-1,1), (j_2)} \right) \frac{n(n-1)}{2n-1} \]

\[ - \frac{2}{r_1^2} \frac{\partial}{\partial j} \left( e^{-ij} \frac{\partial}{\partial j} A^{(n-1,1), (j_2)} \right) \frac{n}{2n-1} - \frac{1}{r_1^2} A^{(n-1,1), (j_2)} \frac{n}{2n-1} + \frac{1}{r_1^2} e^{-ij} \frac{\partial}{\partial j} A^{(n-1,1), (j_2)} \frac{n(n-1)}{2n-1} \]

\[ + \frac{1}{r_1^2} e^{-ij} A^{(n-1,1), (j_2)} \frac{n^2(n-1)}{2n-1} + \frac{4}{r_1^2} e^{-ij} \frac{\partial}{\partial j} A^{(n-1,1), (j_2)} \frac{n^2(n-1)}{2n-1} , \]
\[ d(2, 6) = -\frac{1}{r_i^2} \frac{\partial}{\partial j} A_{n,3}^{(n-1), r}(j_i) n(n + 1) + \frac{1}{r_i^2} \frac{\partial}{\partial j} \left( e^{-r_i A_{n,2}^{(n-1), r}(j_i)} \right) \frac{n(n - 1)}{2n - 1} \]
\[ -\frac{1}{r_i^2} \frac{\partial}{\partial j} \left( e^{-r_i A_{n-1,1}^{(n-1), r}(j_i)} \right) \frac{n(n - 1)}{2n - 1} - \frac{1}{r_i^2} \frac{\partial}{\partial j} \left( e^{-r_i A_{n-1,2}^{(n-1), r}(j_i)} \right) \frac{n(n - 1)}{2n - 1} \]
\[ -\frac{2}{r_i^2} \frac{\partial}{\partial j} \left( e^{-r_i A_{n-1,1}^{(n-1), r}(j_i)} \right) \frac{n(n - 1)}{2n - 1} - \frac{2}{r_i^2} \frac{\partial}{\partial j} \left( e^{-r_i A_{n-1,2}^{(n-1), r}(j_i)} \right) \frac{n(n - 1)}{2n - 1} \]
\[ + \frac{1}{r_i^2} e^{-r_i A_{n-1,1}^{(n-1), r}(j_i)} \frac{n^2(n - 1)}{2n - 1} - \frac{2}{r_i^2} e^{-r_i A_{n-1,2}^{(n-1), r}(j_i)} \frac{n^2(n - 1)}{2n - 1} \]
\[ + \frac{1}{r_i^2} e^{-r_i A_{n-1,1}^{(n-1), r}(j_i)} \frac{n^2(n - 1)}{2n - 1} + \frac{4}{r_i^2} e^{-r_i A_{n-1,2}^{(n-1), r}(j_i)} \frac{n^2(n - 1)}{2n - 1} \]

\[ d(3, 1) = 0, \quad d(3, 2) = 0, \]

\[ d(3, 3) = -\frac{3}{r_2} e^{-r_i A_{n-1,1}^{(n-1), r}(j_2)} \frac{n(n - 1)}{2n - 1} - \frac{3}{2} k^2 r_2 e^{-r_i A_{n-1,1}^{(n-1), r}(j_2)} \frac{1}{2n - 1} + \]
\[ \frac{1}{r_2} e^{-r_i A_{n-1,1}^{(n-1), r}(j_2)} \frac{n(n - 1)}{2n - 1} - \frac{1}{r_2} \frac{\partial}{\partial j} A_{n-1,1}^{(n-1), r}(j_2) - \frac{1}{r_2} A_{n-1,1}^{(n-1), r}(j_2) \]
\[ -\frac{2}{r_2} e^{-r_i A_{n-1,1}^{(n-1), r}(j_2)} \frac{1}{2n - 1} + \frac{1}{r_2} e^{-r_i A_{n-1,1}^{(n-1), r}(j_2)} \frac{n - 1}{2n - 1} \]

\[ d(3, 4) = -\frac{3}{r_2} e^{-r_i A_{n-1,2}^{(n-1), r}(j_2)} \frac{n(n - 1)}{2n - 1} - \frac{3}{2} k^2 r_2 e^{-r_i A_{n-1,2}^{(n-1), r}(j_2)} \frac{1}{2n - 1} + \]
\[ \frac{1}{r_2} e^{-r_i A_{n-1,2}^{(n-1), r}(j_2)} \frac{n(n - 1)}{2n - 1} - \frac{1}{r_2} \frac{\partial}{\partial j} A_{n-1,2}^{(n-1), r}(j_2) - \frac{1}{r_2} A_{n-1,2}^{(n-1), r}(j_2) \]
\[ -\frac{2}{r_2} e^{-r_i A_{n-1,2}^{(n-1), r}(j_2)} \frac{1}{2n - 1} + \frac{1}{r_2} e^{-r_i A_{n-1,2}^{(n-1), r}(j_2)} \frac{n - 1}{2n - 1} \]

\[ d(3, 5) = 0, \quad d(3, 6) = 0, \]

\[ d(4, 1) = 0, \quad d(4, 2) = 0, \]

\[ d(4, 3) = -\frac{3}{r_1} e^{-r_i A_{n-1,1}^{(n-1), r}(j_1)} \frac{n(n - 1)}{2n - 1} - \frac{3}{2} k^2 r_1 e^{-r_i A_{n-1,1}^{(n-1), r}(j_1)} \frac{1}{2n - 1} + \]
\[ \frac{1}{r_1} e^{-r_i A_{n-1,1}^{(n-1), r}(j_1)} \frac{n(n - 1)}{2n - 1} - \frac{1}{r_1} \frac{\partial}{\partial j} A_{n-1,1}^{(n-1), r}(j_1) - \frac{1}{r_1} A_{n-1,1}^{(n-1), r}(j_1) \]
\[ -\frac{2}{r_1} e^{-r_i A_{n-1,1}^{(n-1), r}(j_1)} \frac{1}{2n - 1} + \frac{1}{r_1} e^{-r_i A_{n-1,1}^{(n-1), r}(j_1)} \frac{n - 1}{2n - 1} \]
\[ d(4,4) = -\frac{3}{r_1} e^{-\lambda} \frac{\partial}{\partial j} A_{n=1,1}^{(n-1),s} (j_1) \frac{n-1}{2n-1} - \frac{3}{2} k_2^2 r_1 e^{-\lambda} A_{n=1,2}^{(n-1),s} (j_1) \frac{1}{2n-1} + \]

\[ \frac{1}{r_1} e^{-\lambda} A_{n=1,2}^{(n-1),s} (j_1) \frac{n(n-1)}{2n-1} - \frac{1}{r_1} \frac{\partial}{\partial j} A_{n=1,2}^{(n-1),s} (j_1) - \frac{1}{r_1} A_{n=1,2}^{(n-1),s} (j_1) \]

\[ - \frac{2}{r_1} e^{-\lambda} A_{n=1,2}^{(n-1),s} (j_1) \frac{1}{2n-1} + \frac{1}{r_1} e^{-\lambda} A_{n=1,2}^{(n-1),s} (j_1) \frac{n-1}{2n-1} , \]

\[ d(5,5) = 0, \quad d(4,6) = 0 , \]

\[ d(5,1) = \frac{2}{r_2} e^{-\lambda} \frac{\partial}{\partial j} A_{n=1,1}^{(n-1),p} (j_2) \frac{1}{2n-1} - \frac{1}{r_2} \left( \frac{\partial}{\partial j} A_{n=1,1}^{(n-1),p} (j_2) + A_{n=1,1}^{(n-1),p} (j_2) \right) \]

\[ + \frac{2}{r_2} e^{-\lambda} \left( 2 \frac{\partial}{\partial j} A_{n=1,1}^{(n-1),p} (j_2) + A_{n=1,1}^{(n-1),p} (j_2) \right) \frac{n-1}{2n-1} , \]

\[ d(5,2) = \frac{2}{r_2} e^{-\lambda} \frac{\partial}{\partial j} A_{n=1,2}^{(n-1),p} (j_2) \frac{1}{2n-1} - \frac{1}{r_2} \left( \frac{\partial}{\partial j} A_{n=1,2}^{(n-1),p} (j_2) + A_{n=1,2}^{(n-1),p} (j_2) \right) \]

\[ + \frac{2}{r_2} e^{-\lambda} \left( 2 \frac{\partial}{\partial j} A_{n=1,2}^{(n-1),p} (j_2) + A_{n=1,2}^{(n-1),p} (j_2) \right) \frac{n-1}{2n-1} , \]

\[ d(5,3) = 0, \quad d(5,4) = 0 , \]

\[ d(5,5) = \frac{1}{r_2} \frac{\partial^2}{\partial j^2} A_{n=1,1}^{(n-1),s} (j_2) - \frac{1}{r_2} \frac{\partial^2}{\partial j^2} \left( e^{-\lambda} A_{n=1,1}^{(n-1),s} (j_2) \right) \frac{n-1}{2n-1} \]

\[ + \frac{1}{r_2} \frac{\partial^2}{\partial j^2} \left( e^{-\lambda} \frac{\partial}{\partial j} A_{n=1,1}^{(n-1),s} (j_2) \right) \frac{1}{2n-1} - \frac{4}{r_2} e^{-\lambda} \frac{\partial^2}{\partial j^2} A_{n=1,1}^{(n-1),s} (j_2) \frac{n-1}{2n-1} \]

\[ + \frac{k_2}{2} \frac{\partial}{\partial j} A_{n=1,1}^{(n-1),s} (j_2) + \frac{k_2}{2} e^{-\lambda} A_{n=1,1}^{(n-1),s} (j_2) \frac{n-1}{2n-1} - \frac{k_2}{2} e^{-\lambda} \frac{\partial}{\partial j} A_{n=1,1}^{(n-1),s} (j_2) \frac{1}{2n-1} \]

\[ - \frac{1}{r_2} A_{n=1,1}^{(n-1),s} (j_2) + \frac{1}{r_2} e^{-\lambda} A_{n=1,1}^{(n-1),s} (j_2) \frac{n-1}{2n-1} \]

\[ - \frac{1}{r_2} e^{-\lambda} \frac{\partial}{\partial j} A_{n=1,1}^{(n-1),s} (j_2) \frac{1}{2n-1} + \frac{2}{r_2} e^{-\lambda} A_{n=1,1}^{(n-1),s} (j_2) \frac{n-1}{2n-1} \]

\[ - \frac{2}{r_2} e^{-\lambda} A_{n=1,1}^{(n-1),s} (j_2) \frac{1}{2n-1} + \frac{2}{r_2} e^{-\lambda} A_{n=1,1}^{(n-1),s} (j_2) \frac{n-1}{2n-1} , \]
\[
\begin{align*}
    d(5,6) &= \frac{1}{r_2^2} \frac{\partial^2}{\partial j^2} A^{(n-1),x}_{n-2} (j_2) - \frac{1}{r_2^2} \frac{\partial^2}{\partial j^2} (e^{-j_2} A^{(n-1),x}_{n-1,2} (j_2)) \frac{n-1}{2n-1} \\
    &+ \frac{1}{r_2^2} \frac{\partial^2}{\partial j^2} \left( e^{-j_2} \frac{\partial}{\partial j} A^{(n-1),x}_{n-1,2} (j_2) \right) \frac{1}{2n-1} - \frac{4}{r_2^2} e^{-j_2} \frac{\partial^2}{\partial j^2} A^{(n-1),x}_{n-1,2} (j_2) \frac{n-1}{2n-1} \\
    &+ \frac{k_2^2}{2} A^{(n-1),x}_{n-2} (j_2) + \frac{k_2^2}{2} e^{-j_2} A^{(n-1),x}_{n-1,2} (j_2) \frac{n-1}{2n-1} - \frac{k_2^2}{2} e^{-j_2} \frac{\partial}{\partial j} A^{(n-1),x}_{n-1,2} (j_2) \frac{1}{2n-1} \\
    &- \frac{1}{r_2^2} A^{(n-1),x}_{n-2} (j_2) + \frac{1}{r_2^2} e^{-j_2} A^{(n-1),x}_{n-1,2} (j_2) \frac{n-1}{2n-1} \\
    &- \frac{1}{r_2^2} e^{-j_2} \frac{\partial}{\partial j} A^{(n-1),x}_{n-1,2} (j_2) \frac{1}{2n-1} + \frac{2}{r_2^2} e^{-j_2} A^{(n-1),x}_{n-1,2} (j_2) \frac{n-1}{2n-1} \\
    &- \frac{2}{r_2^2} e^{-j_2} A^{(n-1),x}_{n-1,2} (j_2) \frac{1}{2n-1} + \frac{2}{r_2^2} e^{-j_2} A^{(n-1),x}_{n-1,2} (j_2) \frac{n-1}{2n-1},
\end{align*}
\]

\[
\begin{align*}
    d(6,1) &= \frac{2}{r_1^2} e^{-j_1} \frac{\partial}{\partial j} A^{(n-1),x}_{n-2,1} (j_1) \frac{1}{2n-1} - \frac{1}{r_1^2} \left( \frac{\partial}{\partial j} A^{(n-1),x}_{n-1,1} (j_1) + A^{(n-1),x}_{n-1,1} (j_1) \right) \\
    &+ \frac{2}{r_1^2} e^{-j_1} \left( \frac{\partial}{\partial j} A^{(n-1),x}_{n-1,1} (j_1) + A^{(n-1),x}_{n-1,1} (j_1) \right) \frac{n-1}{2n-1},
\end{align*}
\]

\[
\begin{align*}
    d(6,2) &= \frac{2}{r_1^2} e^{-j_1} \frac{\partial}{\partial j} A^{(n-1),x}_{n-1,2} (j_1) \frac{1}{2n-1} - \frac{1}{r_1^2} \left( \frac{\partial}{\partial j} A^{(n-1),x}_{n-1,1} (j_1) + A^{(n-1),x}_{n-1,1} (j_1) \right) \\
    &+ \frac{2}{r_1^2} e^{-j_1} \left( \frac{\partial}{\partial j} A^{(n-1),x}_{n-1,1} (j_1) + A^{(n-1),x}_{n-1,1} (j_1) \right) \frac{n-1}{2n-1},
\end{align*}
\]

\[
\begin{align*}
    d(6,3) &= 0, \quad d(6,4) = 0,
\end{align*}
\]

\[
\begin{align*}
    d(6,5) &= \frac{1}{r_1^2} \frac{\partial^2}{\partial j^2} A^{(n-1),x}_{n-1,1} (j_1) - \frac{1}{r_1^2} \frac{\partial^2}{\partial j^2} (e^{-j_1} A^{(n-1),x}_{n-2,1} (j_1)) \frac{n-1}{2n-1} \\
    &+ \frac{1}{r_1^2} \frac{\partial^2}{\partial j^2} \left( e^{-j_1} \frac{\partial}{\partial j} A^{(n-1),x}_{n-2,1} (j_1) \right) \frac{1}{2n-1} - \frac{4}{r_1^2} e^{-j_1} \frac{\partial^2}{\partial j^2} A^{(n-1),x}_{n-2,1} (j_1) \frac{n-1}{2n-1} \\
    &+ \frac{k_1^2}{2} A^{(n-1),x}_{n-1,1} (j_1) + \frac{k_1^2}{2} e^{-j_1} A^{(n-1),x}_{n-2,1} (j_1) \frac{n-1}{2n-1} - \frac{k_1^2}{2} e^{-j_1} \frac{\partial}{\partial j} A^{(n-1),x}_{n-2,1} (j_1) \frac{1}{2n-1} \\
    &- \frac{1}{r_1^2} A^{(n-1),x}_{n-1,1} (j_1) + \frac{1}{r_1^2} e^{-j_1} A^{(n-1),x}_{n-1,1} (j_1) \frac{n-1}{2n-1} \\
    &- \frac{1}{r_1^2} e^{-j_1} \frac{\partial}{\partial j} A^{(n-1),x}_{n-1,1} (j_1) \frac{1}{2n-1} + \frac{2}{r_1^2} e^{-j_1} A^{(n-1),x}_{n-1,1} (j_1) \frac{n-1}{2n-1} \\
    &- \frac{2}{r_1^2} e^{-j_1} A^{(n-1),x}_{n-1,1} (j_1) \frac{1}{2n-1} + \frac{2}{r_1^2} e^{-j_1} A^{(n-1),x}_{n-1,1} (j_1) \frac{n-1}{2n-1}.
\end{align*}
\]
\[ d(6,6) = \frac{1}{r_1^2} \frac{\partial^2}{\partial j^2} A_{n-1,2}^{(n-1),s}(j_1) - \frac{1}{r_1^2} \frac{\partial^2}{\partial j^2} \left( e^{-j_1} A_{n-1,2}^{(n-1),s}(j_1) \right) \frac{n-1}{2n-1} \]
\[ + \frac{1}{r_1^2} \frac{\partial^2}{\partial j^2} \left( e^{-j_1} \frac{\partial}{\partial j} A_{n-1,2}^{(n-1),s}(j_1) \right) \frac{1}{2n-1} - \frac{4}{r_1^2} \frac{1}{2n-1} \]
\[ + \frac{k^2}{r_1^2} A_{n-1,2}^{(n-1),s}(j_1) + \frac{k^2}{2} e^{-j_1} A_{n-1,2}^{(n-1),s}(j_1) \frac{n-1}{2n-1} + \frac{k^2}{2} e^{-j_1} \frac{\partial}{\partial j} A_{n-1,2}^{(n-1),s}(j_1) \frac{1}{2n-1} \]
\[ - \frac{1}{r_1^2} A_{n-1,2}^{(n-1),s}(j_1) + \frac{1}{r_1^2} \frac{\partial}{\partial j} A_{n-1,2}^{(n-1),s}(j_1) \frac{n-1}{2n-1} \]
\[ - \frac{1}{r_1^2} e^{-j_1} \frac{\partial}{\partial j} A_{n-1,2}^{(n-1),s}(j_1) \frac{1}{2n-1} + \frac{2}{r_1^2} e^{-j_1} A_{n-1,2}^{(n-1),s}(j_1) \frac{n-1}{2n-1} \]
\[ - \frac{2}{r_1^2} e^{-j_1} A_{n-1,2}^{(n-1),s}(j_1) \frac{1}{2n-1} + \frac{2}{r_1^2} e^{-j_1} A_{n-1,2}^{(n-1),s}(j_1) \frac{n-1}{2n-1} \]

**APPENDIX C:**

Elements of the upper diagonal block matrices \( D_{n+1}^{(n)} \):

\[ d(1,1) = -k_p^2 A_{n,1}^{(n+1),p}(j_2) - \frac{4}{r_2} e^{-j_2} \left( \frac{\partial}{\partial j} A_{n,1}^{(n+1),p}(j_2) + (n+1)(n+2) A_{n,1}^{(n+1),p}(j_2) \right) \frac{n+1}{2n+3} \]
\[ + \frac{1}{r_2^2} \left( 2 \frac{\partial}{\partial j} A_{n,1}^{(n+1),p}(j_2) + n(n+1) A_{n,1}^{(n+1),p}(j_2) \right) - \frac{\lambda k_p^2}{2} A_{n,1}^{(n+1),p}(j_2), \]
\[ d(1,2) = -k_p^2 A_{n,2}^{(n+1),p}(j_2) - \frac{4}{r_2} e^{-j_2} \left( \frac{\partial}{\partial j} A_{n,2}^{(n+1),p}(j_2) + (n+1)(n+2) A_{n,2}^{(n+1),p}(j_2) \right) \frac{n+1}{2n+3} \]
\[ + \frac{1}{r_2^2} \left( 2 \frac{\partial}{\partial j} A_{n,2}^{(n+1),p}(j_2) + n(n+1) A_{n,2}^{(n+1),p}(j_2) \right) - \frac{\lambda k_p^2}{2} A_{n,2}^{(n+1),p}(j_2), \]
\[ d(1,3) = d(1,4) = 0, \]
\[ d(1.5) = -\frac{1}{r_2^2} \frac{\partial}{\partial j} A_{n+1,1}^{(n+1),\sigma}(j_2)n(n+1) + \frac{1}{r_2^2} \frac{\partial}{\partial j} \left( e^{-j_2} A_{n+1,1}^{(n+1),\sigma}(j_2) \right) \frac{(n+1)(n+2)}{2n+3} \]
\[ + \frac{2}{r_2^2} e^{-j_2} \frac{\partial}{\partial j} A_{n+1,1}^{(n+1),\sigma}(j_2) \frac{(n+1)(n+2)}{2n+3} - \frac{1}{r_2^2} A_{n+1,1}^{(n+1),\sigma}(j_2)n(n+1) \]
\[ - \frac{2}{r_2^2} e^{-j_2} \frac{\partial}{\partial j} A_{n+1,1}^{(n+1),\sigma}(j_2) \frac{n+1}{2n+3} - \frac{1}{r_2^2} e^{-j_2} \frac{\partial}{\partial j} A_{n+1,1}^{(n+1),\sigma}(j_2) \frac{(n+1)(n+2)}{2n+3} \]
\[ + \frac{2}{r_2^2} e^{-j_2} A_{n+1,1}^{(n+1),\sigma}(j_2) \frac{(n+1)^2(n+2)}{2n+3} + \frac{1}{r_2^2} \frac{\partial}{\partial j} \left( e^{-j_2} \frac{\partial}{\partial j} A_{n+1,1}^{(n+1),\sigma}(j_2) \right) \frac{n(n+1)}{2n+3} \]
\[ + \frac{4}{r_2^2} e^{-j_2} \frac{\partial}{\partial j} A_{n+1,1}^{(n+1),\sigma}(j_2) \frac{(n+1)^2(n+2)}{2n+3}, \]

\[ d(1.6) = -\frac{1}{r_2^2} \frac{\partial}{\partial j} A_{n+1,2}^{(n+1),\sigma}(j_2)n(n+1) + \frac{1}{r_2^2} \frac{\partial}{\partial j} \left( e^{-j_2} A_{n+1,2}^{(n+1),\sigma}(j_2) \right) \frac{(n+1)(n+2)}{2n+3} \]
\[ + \frac{2}{r_2^2} e^{-j_2} \frac{\partial}{\partial j} A_{n+1,2}^{(n+1),\sigma}(j_2) \frac{(n+1)^2(n+2)}{2n+3} - \frac{1}{r_2^2} A_{n+1,2}^{(n+1),\sigma}(j_2)n(n+1) \]
\[ - \frac{2}{r_2^2} e^{-j_2} \frac{\partial}{\partial j} A_{n+1,2}^{(n+1),\sigma}(j_2) \frac{n+1}{2n+3} - \frac{1}{r_2^2} e^{-j_2} \frac{\partial}{\partial j} A_{n+1,2}^{(n+1),\sigma}(j_2) \frac{(n+1)(n+2)}{2n+3} \]
\[ + \frac{2}{r_2^2} e^{-j_2} A_{n+1,2}^{(n+1),\sigma}(j_2) \frac{(n+1)^2(n+2)}{2n+3} + \frac{1}{r_2^2} \frac{\partial}{\partial j} \left( e^{-j_2} \frac{\partial}{\partial j} A_{n+1,2}^{(n+1),\sigma}(j_2) \right) \frac{n(n+1)}{2n+3} \]
\[ + \frac{4}{r_2^2} e^{-j_2} \frac{\partial}{\partial j} A_{n+1,2}^{(n+1),\sigma}(j_2) \frac{(n+1)^2(n+2)}{2n+3}, \]

\[ d(2.1) = -k_p^2 A_{n+1}^{(n+1),p}(j_1) - \frac{4}{r_1^2} e^{-j_1} \left( \frac{\partial}{\partial j} A_{n+1,1}^{(n+1),p}(j_1) + (n+1)(n+2) A_{n+1,1}^{(n+1),p}(j_1) \right) \frac{n+1}{2n+3} \]
\[ + \frac{1}{r_1^2} \left( 2 \frac{\partial}{\partial j} A_{n+1,1}^{(n+1),p}(j_1) + n(n+1) A_{n+1,1}^{(n+1),p}(j_1) \right) - \frac{\lambda k_p^2}{2} A_{n+1}^{(n+1),p}(j_1), \]

\[ d(2.2) = -k_p^2 A_{n+1,2}^{(n+1),p}(j_1) - \frac{4}{r_1^2} e^{-j_1} \left( \frac{\partial}{\partial j} A_{n+1,2}^{(n+1),p}(j_1) + (n+1)(n+2) A_{n+1,2}^{(n+1),p}(j_1) \right) \frac{n+1}{2n+3} \]
\[ + \frac{1}{r_1^2} \left( 2 \frac{\partial}{\partial j} A_{n+1,2}^{(n+1),p}(j_1) + n(n+1) A_{n+1,2}^{(n+1),p}(j_1) \right) - \frac{\lambda k_p^2}{2} A_{n+1}^{(n+1),p}(j_1), \]

\[ d(2.3) = 0, \ d(2.4) = 0, \]

- 42 -
\[d(2,5) = -\frac{1}{r_1^3} \frac{\partial}{\partial j} A_{n,1}^{(\kappa+1),s}(j_1) n(n+1) + \frac{1}{r_1^3} \frac{\partial}{\partial j} \left( e^{-j} A_{n,1}^{(\kappa+1),s}(j_1) \right) \frac{(n+1)(n+2)}{2n+3}
- \frac{1}{r_1^3} \frac{\partial}{\partial j} A_{n,1}^{(\kappa+1),s}(j_1) \frac{(n+1)^2(n+2)}{2n+3} - \frac{1}{r_1^2} A_{n,1}^{(\kappa+1),s}(j_1) n(n+1)
- \frac{1}{r_1} e^{-j} A_{n,1}^{(\kappa+1),s}(j_1) \frac{(n+1)(n+2)}{2n+3} + \frac{1}{r_1} e^{-j} A_{n,1}^{(\kappa+1),s}(j_1) \frac{(n+1)^2(n+2)}{2n+3}
- \frac{2}{r_1} e^{-j} \frac{\partial}{\partial j} A_{n,1}^{(\kappa+1),s}(j_1) \frac{n+1}{2n+3} - \frac{1}{r_1} e^{-j} \frac{\partial}{\partial j} A_{n,1}^{(\kappa+1),s}(j_1) \frac{(n+1)(n+2)}{2n+3}
+ \frac{4}{r_1} e^{-j} \frac{\partial}{\partial j} A_{n,1}^{(\kappa+1),s}(j_1) \frac{(n+1)^2(n+2)}{2n+3},
\]

\[d(2,6) = -\frac{1}{r_1^3} \frac{\partial}{\partial j} A_{n,2}^{(\kappa+1),s}(j_1) n(n+1) + \frac{1}{r_1^3} \frac{\partial}{\partial j} \left( e^{-j} A_{n,2}^{(\kappa+1),s}(j_1) \right) \frac{(n+1)(n+2)}{2n+3}
+ \frac{1}{r_1^3} \frac{\partial}{\partial j} \left( e^{-j} A_{n,2}^{(\kappa+1),s}(j_1) \right) \frac{(n+1)^2(n+2)}{2n+3} - \frac{1}{r_1^2} A_{n,2}^{(\kappa+1),s}(j_1) n(n+1)
+ \frac{1}{r_1} e^{-j} A_{n,2}^{(\kappa+1),s}(j_1) \frac{(n+1)(n+2)}{2n+3} + \frac{1}{r_1} e^{-j} A_{n,2}^{(\kappa+1),s}(j_1) \frac{(n+1)^2(n+2)}{2n+3}
- \frac{2}{r_1} e^{-j} \frac{\partial}{\partial j} A_{n,2}^{(\kappa+1),s}(j_1) \frac{n+1}{2n+3} - \frac{1}{r_1} e^{-j} \frac{\partial}{\partial j} A_{n,2}^{(\kappa+1),s}(j_1) \frac{(n+1)(n+2)}{2n+3}
+ \frac{4}{r_1} e^{-j} \frac{\partial}{\partial j} A_{n,2}^{(\kappa+1),s}(j_1) \frac{(n+1)^2(n+2)}{2n+3},
\]

\[d(3,1) = 0, \ d(3,2) = 0,
\]

\[d(3,3) = -\frac{3}{r_2} e^{-j_2} \frac{\partial}{\partial j_2} A_{n,3}^{(\kappa+1),s}(j_2) \frac{n+2}{2n+3} + \frac{3k_2}{2} e^{-j_2} A_{n,3}^{(\kappa+1),s}(j_2) \frac{1}{2n+3}
- \frac{1}{r_2} e^{-j} A_{n,3}^{(\kappa+1),s}(j_2) \frac{(n+2)(n+1)}{2n+3} - \frac{1}{r_2} e^{-j} A_{n,3}^{(\kappa+1),s}(j_2)
+ \frac{1}{r_2} e^{-j} A_{n,3}^{(\kappa+1),s}(j_2) \frac{n+2}{2n+3} - \frac{1}{r_2} A_{n,3}^{(\kappa+1),s}(j_2) + \frac{2}{r_2} e^{-j} A_{n,3}^{(\kappa+1),s}(j_2) \frac{1}{2n+3},
\]
\[ d(3,4) = -\frac{3}{r_2} \cdot e^{-\frac{j_2}{r_2}} \cdot \frac{\partial}{\partial j_2} A_{x_{n+1,2}}^{(n+1),p}(j_2) \cdot \frac{n+2}{2n+3} + \frac{3}{2} k_2 r_2 e^{-\frac{j_2}{r_2}} A_{x_{n+1,2}}^{(n+1),p}(j_2) \cdot \frac{1}{2n+3} \]

\[ - \frac{1}{r_2} e^{-\frac{j_2}{r_2}} A_{x_{n+1,2}}^{(n+1),p}(j_2) \cdot \frac{(n+2)(n+1)}{2n+3} - \frac{1}{r_2} \cdot \frac{\partial}{\partial j_2} A_{x_{n+1,2}}^{(n+1),p}(j_2) \]

\[ + \frac{1}{r_2} e^{-\frac{j_2}{r_2}} A_{x_{n+1,2}}^{(n+1),p}(j_2) \cdot \frac{n+2}{2n+3} - \frac{1}{r_2} A_{x_{n+1,2}}^{(n+1),p}(j_2) + \frac{2}{r_2} e^{-\frac{j_2}{r_2}} A_{x_{n+1,2}}^{(n+1),p}(j_2) \cdot \frac{1}{2n+3}, \]

\[ d(3,5) = 0, \quad d(3,6) = 0, \]

\[ d(4,1) = 0, \quad d(4,2) = 0, \]

\[ d(4,3) = -\frac{3}{r_1} \cdot e^{-\frac{j_1}{r_1}} \cdot \frac{\partial}{\partial j_1} A_{x_{n+1,1}}^{(n+1),p}(j_1) \cdot \frac{n+2}{2n+3} + \frac{3}{2} k_1 r_1 e^{-\frac{j_1}{r_1}} A_{x_{n+1,1}}^{(n+1),p}(j_1) \cdot \frac{1}{2n+3} \]

\[ - \frac{1}{r_1} e^{-\frac{j_1}{r_1}} A_{x_{n+1,1}}^{(n+1),p}(j_1) \cdot \frac{(n+2)(n+1)}{2n+3} - \frac{1}{r_1} \cdot \frac{\partial}{\partial j_1} A_{x_{n+1,1}}^{(n+1),p}(j_1) \]

\[ + \frac{1}{r_1} e^{-\frac{j_1}{r_1}} A_{x_{n+1,1}}^{(n+1),p}(j_1) \cdot \frac{n+2}{2n+3} - \frac{1}{r_1} A_{x_{n+1,1}}^{(n+1),p}(j_1) + \frac{2}{r_1} e^{-\frac{j_1}{r_1}} A_{x_{n+1,1}}^{(n+1),p}(j_1) \cdot \frac{1}{2n+3}, \]

\[ d(4,4) = -\frac{3}{r_1} \cdot e^{-\frac{j_1}{r_1}} \cdot \frac{\partial}{\partial j_1} A_{x_{n+1,1}}^{(n+1),p}(j_1) \cdot \frac{n+2}{2n+3} + \frac{3}{2} k_1 r_1 e^{-\frac{j_1}{r_1}} A_{x_{n+1,1}}^{(n+1),p}(j_1) \cdot \frac{1}{2n+3} \]

\[ - \frac{1}{r_1} e^{-\frac{j_1}{r_1}} A_{x_{n+1,1}}^{(n+1),p}(j_1) \cdot \frac{(n+2)(n+1)}{2n+3} - \frac{1}{r_1} \cdot \frac{\partial}{\partial j_1} A_{x_{n+1,1}}^{(n+1),p}(j_1) \]

\[ + \frac{1}{r_1} e^{-\frac{j_1}{r_1}} A_{x_{n+1,1}}^{(n+1),p}(j_1) \cdot \frac{n+2}{2n+3} - \frac{1}{r_1} A_{x_{n+1,1}}^{(n+1),p}(j_1) + \frac{2}{r_1} e^{-\frac{j_1}{r_1}} A_{x_{n+1,1}}^{(n+1),p}(j_1) \cdot \frac{1}{2n+3}, \]

\[ d(4,5) = 0, \quad d(4,6) = 0, \]

\[ d(5,1) = -\frac{2}{r_2} \cdot e^{-\frac{j_2}{r_2}} \cdot \frac{\partial}{\partial j_2} A_{x_{n+1,1}}^{(n+1),p}(j_2) \cdot \frac{1}{2n+3} - \frac{1}{r_2} \left( \frac{\partial}{\partial j_2} A_{x_{n+1,1}}^{(n+1),p}(j_2) + A_{x_{n+1,1}}^{(n+1),p}(j_2) \right) \]

\[ + \frac{2}{r_2} e^{-\frac{j_2}{r_2}} \left( \frac{\partial}{\partial j_2} A_{x_{n+1,1}}^{(n+1),p}(j_2) + A_{x_{n+1,1}}^{(n+1),p}(j_2) \right) \cdot \frac{n+2}{2n+3}, \]

\[ d(5,2) = -\frac{2}{r_2} \cdot e^{-\frac{j_2}{r_2}} \cdot \frac{\partial}{\partial j_2} A_{x_{n+1,2}}^{(n+1),p}(j_2) \cdot \frac{1}{2n+3} - \frac{1}{r_2} \left( \frac{\partial}{\partial j_2} A_{x_{n+1,2}}^{(n+1),p}(j_2) + A_{x_{n+1,2}}^{(n+1),p}(j_2) \right) \]

\[ + \frac{2}{r_2} e^{-\frac{j_2}{r_2}} \left( \frac{\partial}{\partial j_2} A_{x_{n+1,2}}^{(n+1),p}(j_2) + A_{x_{n+1,2}}^{(n+1),p}(j_2) \right) \cdot \frac{n+2}{2n+3}, \]

\[ d(5,3) = 0, \quad d(5,4) = 0, \]
\[
d(5,5) = \frac{1}{r_2^2} \frac{\partial^2}{\partial j^2} A_{n=1}^{(n+1),r}(j_2) - \frac{1}{r_2^2} \frac{\partial^2}{\partial j^2} (e^{-j_1} A_{n=1}^{(n+1),r}(j_2)) \frac{n+2}{2n+3}
- \frac{1}{r_2^2} \frac{\partial^2}{\partial j^2} (e^{-j_1} A_{n=1}^{(n+1),r}(j_2)) \frac{1}{2n+3} - \frac{4}{r_2^2} e^{-j_1} \frac{\partial^2}{\partial j^2} A_{n=1}^{(n+1),r}(j_2) \frac{n+2}{2n+3}
+ \frac{k_2}{2} A_{n=1}^{(n+1),r}(j_2) + \frac{k_2}{2} e^{-j_1} A_{n=1}^{(n+1),r}(j_2) \frac{n+2}{2n+3} + \frac{k_2}{2} e^{-j_1} \frac{\partial}{\partial j} A_{n=1}^{(n+1),r}(j_2) \frac{1}{2n+3}
- \frac{1}{r_2^2} A_{n=1}^{(n+1),r}(j_2) + \frac{1}{2n+3} e^{-j_1} A_{n=1}^{(n+1),r}(j_2) \frac{1}{2n+3} + \frac{1}{r_2^2} \frac{\partial}{\partial j} A_{n=1}^{(n+1),r}(j_2) \frac{1}{2n+3}
+ \frac{2}{r_2^2} e^{-j_1} A_{n=1}^{(n+1),r}(j_2) \frac{n+2}{2n+3} + \frac{2}{r_2^2} e^{-j_1} A_{n=1}^{(n+1),r}(j_2) \frac{1}{2n+3} + \frac{2}{r_2^2} e^{-j_1} A_{n=1}^{(n+1),r}(j_2) \frac{n+2}{2n+3}.
\]

\[
d(5,6) = \frac{1}{r_2^2} \frac{\partial^2}{\partial j^2} A_{n=2}^{(n+1),r}(j_2) - \frac{1}{r_2^2} \frac{\partial^2}{\partial j^2} (e^{-j_1} A_{n=2}^{(n+1),r}(j_2)) \frac{n+2}{2n+3}
- \frac{1}{r_2^2} \frac{\partial^2}{\partial j^2} (e^{-j_1} A_{n=2}^{(n+1),r}(j_2)) \frac{1}{2n+3} - \frac{4}{r_2^2} e^{-j_1} \frac{\partial^2}{\partial j^2} A_{n=2}^{(n+1),r}(j_2) \frac{n+2}{2n+3}
+ \frac{k_2}{2} A_{n=2}^{(n+1),r}(j_2) + \frac{k_2}{2} e^{-j_1} A_{n=2}^{(n+1),r}(j_2) \frac{n+2}{2n+3} + \frac{k_2}{2} e^{-j_1} \frac{\partial}{\partial j} A_{n=2}^{(n+1),r}(j_2) \frac{1}{2n+3}
- \frac{1}{r_2^2} A_{n=2}^{(n+1),r}(j_2) + \frac{1}{2n+3} e^{-j_1} A_{n=2}^{(n+1),r}(j_2) \frac{1}{2n+3} + \frac{1}{r_2^2} \frac{\partial}{\partial j} A_{n=2}^{(n+1),r}(j_2) \frac{1}{2n+3}
+ \frac{2}{r_2^2} e^{-j_1} A_{n=2}^{(n+1),r}(j_2) \frac{n+2}{2n+3} + \frac{2}{r_2^2} e^{-j_1} A_{n=2}^{(n+1),r}(j_2) \frac{1}{2n+3} + \frac{2}{r_2^2} e^{-j_1} A_{n=2}^{(n+1),r}(j_2) \frac{n+2}{2n+3}.
\]

\[
d(6,1) = -\frac{2}{r_1^2} e^{-j_1} \frac{\partial}{\partial j} A_{n=1}^{(n+1),r}(j_1) \frac{1}{2n+3} - \frac{1}{r_1^2} \left( \frac{\partial}{\partial j} A_{n=1}^{(n+1),r}(j_1) + A_{n=1}^{(n+1),r}(j_1) \right)
+ \frac{2}{r_1^2} e^{-j_1} \left( 2 \frac{\partial}{\partial j} A_{n=1}^{(n+1),r}(j_1) + A_{n=1}^{(n+1),r}(j_1) \right) \frac{n+2}{2n+3},
\]

\[
d(6,2) = -\frac{2}{r_1^2} e^{-j_1} \frac{\partial}{\partial j} A_{n=2}^{(n+1),r}(j_1) \frac{1}{2n+3} - \frac{1}{r_1^2} \left( \frac{\partial}{\partial j} A_{n=2}^{(n+1),r}(j_1) + A_{n=2}^{(n+1),r}(j_1) \right)
+ \frac{2}{r_1^2} e^{-j_1} \left( 2 \frac{\partial}{\partial j} A_{n=2}^{(n+1),r}(j_1) + A_{n=2}^{(n+1),r}(j_1) \right) \frac{n+2}{2n+3},
\]

\[
d(6,3) = 0, \quad d(6,4) = 0,
\]
\[ d(6, 5) = \frac{1}{r_i^2} \frac{\partial^2}{\partial j^2} A_{n+1, l}^{(n+1), s}(j_i) - \frac{1}{r_i^2} \frac{\partial^2}{\partial j^2} (e^{-i} A_{n+1, l}^{(n+1), s}(j_i)) \frac{n + 2}{2n + 3} \]

\[ - \frac{1}{r_i} \frac{\partial^2}{\partial j^2} (e^{-i} A_{n+1, l}^{(n+1), s}(j_i)) \frac{1}{2n + 3} - \frac{4}{r_i} e^{-i} \frac{\partial^2}{\partial j^2} A_{n+1, l}^{(n+1), s}(j_i) \frac{n + 2}{2n + 3} \]

\[ + \frac{k_f^2}{2} A_{n, l}^{(n+1), s}(j_i) + \frac{k_f^2}{2} e^{-i} A_{n+1, l}^{(n+1), s}(j_i) \frac{n + 2}{2n + 3} + \frac{k_f^2}{2} e^{-i} \frac{\partial}{\partial j} A_{n+1, l}^{(n+1), s}(j_i) \frac{1}{2n + 3} \]

\[ - \frac{1}{r_i} A_{n+1, l}^{(n+1), s}(j_i) + \frac{1}{r_i} e^{-i} A_{n+1, l}^{(n+1), s}(j_i) \frac{n + 2}{2n + 3} + \frac{1}{r_i} e^{-i} \frac{\partial}{\partial j} A_{n+1, l}^{(n+1), s}(j_i) \frac{1}{2n + 3} \]

\[ + \frac{2}{r_i} e^{-i} A_{n+1, l}^{(n+1), s}(j_i) \frac{n + 2}{2n + 3} + \frac{2}{r_i} e^{-i} A_{n+1, l}^{(n+1), s}(j_i) \frac{1}{2n + 3} + \frac{2}{r_i} e^{-i} A_{n+1, l}^{(n+1), s}(j_i) \frac{n + 2}{2n + 3} \]

\[ d(6, 6) = \frac{1}{r_i^2} \frac{\partial^2}{\partial j^2} A_{n+1, 2}^{(n+1), s}(j_i) - \frac{1}{r_i^2} \frac{\partial^2}{\partial j^2} (e^{-i} A_{n+1, 2}^{(n+1), s}(j_i)) \frac{n + 2}{2n + 3} \]

\[ - \frac{1}{r_i} \frac{\partial^2}{\partial j^2} (e^{-i} A_{n+1, 2}^{(n+1), s}(j_i)) \frac{1}{2n + 3} - \frac{4}{r_i} e^{-i} \frac{\partial^2}{\partial j^2} A_{n+1, 2}^{(n+1), s}(j_i) \frac{n + 2}{2n + 3} \]

\[ + \frac{k_f^2}{2} A_{n, 2}^{(n+1), s}(j_i) + \frac{k_f^2}{2} e^{-i} A_{n+1, 2}^{(n+1), s}(j_i) \frac{n + 2}{2n + 3} + \frac{k_f^2}{2} e^{-i} \frac{\partial}{\partial j} A_{n+1, 2}^{(n+1), s}(j_i) \frac{1}{2n + 3} \]

\[ - \frac{1}{r_i} A_{n+1, 2}^{(n+1), s}(j_i) + \frac{1}{r_i} e^{-i} A_{n+1, 2}^{(n+1), s}(j_i) \frac{n + 2}{2n + 3} + \frac{1}{r_i} e^{-i} \frac{\partial}{\partial j} A_{n+1, 2}^{(n+1), s}(j_i) \frac{1}{2n + 3} \]

\[ + \frac{2}{r_i} e^{-i} A_{n+1, 2}^{(n+1), s}(j_i) \frac{n + 2}{2n + 3} + \frac{2}{r_i} e^{-i} A_{n+1, 2}^{(n+1), s}(j_i) \frac{1}{2n + 3} + \frac{2}{r_i} e^{-i} A_{n+1, 2}^{(n+1), s}(j_i) \frac{n + 2}{2n + 3} \]

In Appendices A, B and C we have used for simplicity the convention

\[ \frac{\partial}{\partial x} f(x_i) = \frac{\partial}{\partial x} f(x)|_{x=x_i} . \]