On the Dynamic Characteristics of the Human Skull - Brain System

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THE DYNAMIC CHARACTERISTICS
OF THE HUMAN SKULL - BRAIN SYSTEM

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Abstract

In this work the dynamic characteristics of the human skull-brain system is studied. For the
purpose of our analysis we adopted a model consisted of a hollow sphere (skull), an inviscid and
irrotational fluid (cerebrospinal fluid) and a concentrically located inner elastic sphere (brain). The
mathematical analysis is based on the elasticity solution for the elastic spheres and the simplified
description of the motion of the fluid by the wave equation. The roots of the characteristic equation
were found numerically. The results are in good agreement with other researchers analogous
modelling work, however our three dimensional analysis introduces a new pattern of frequencies to
the natural frequencies spectra of the skull-brain system. The results are compared with
experimental ones and the role of the various system parameters on the natural frequencies is
investigated.

1. INTRODUCTION

The knowledge of how and why the natural frequencies change could aid in the development of a
new medical instrument for the early diagnosis of brain diseases [1]. The experimental
investigations of the resonance frequencies of the human skull have been made on living subjects
as well as cadavers and dry skulls. Khali et al. [2] presented a comprehensive investigation of the
resonance frequencies of the human skull. The resonance frequencies of the human skull in vivo
have been recently investigated by Håkansson et al. [3]. The complexity of the human skull
structure presents an extremely difficult task to one wishing to perform detailed simulation of the
physical processes of the human head by mathematical modelling. For this reason, geometrical
approximations are typically used for analytical investigations. We note that, although finite element
model representations of the head geometry are more desirable, compatible characterisations of the
zonal scalp, skull, dura matter and brain constitutive properties along with engineering definitions of
the associated interface conditions are lacking.

Engin and Liu [4] adopted a model consisting of a thin homogeneous isotropic and spherical shell
containing an inviscid irrotational fluid. Advani and Owings [5] considered a structural modelling of
the human head consisting of a uniform elastic spherical shell (skull) containing an elastic core
(brain). Talhouk and DiMaggio [6] introduced the modelling of a head as a prolate spherical shell
filled with an inviscid liquid. The model of Ref. 6 was used by Miso et al. [7] to study the dynamic
response of a head-neck system to an impulsive load. Recently, Guarino and Elger [8] adopted the
modelling of the human head by a fluid-filled elastic shell containing a concentrically located elastic
sphere. In Ref. 8 an analytical modal analysis was used to obtain the characteristic equation for the
natural frequencies of the physical system considered. The major assumptions adopted by Guarino
and Elger are that the fluid is inviscid, the spherical shell is thin and the modes of the physical
system are axisymmetric.

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The effect of viscosity on the free oscillations of fluid-filled spherical shells was studied by Su [9]. From the results cited in Ref. 9 we obtain the information that the natural frequencies of a vibrating spherical shell will decrease with increase of the viscosity of the contained fluid. The elastomechanical characterisation of brain tissues was studied by Sahay et al. [10]. The model used in Ref. 10 consists of an outer shell representing the rigid cranium and an inner sphere representing the hyperelastic brain. The cranium is supposed to be separated from the brain by a bed of venous fluid. A model which takes into account the viscoelastic nature of the brain tissue would be a more accurate representation of the intracranial complex [11].

In this study we adopt a model for the human head consisting of a hollow sphere (skull) containing a concentrically located elastic sphere (brain). The space between the two spheres is supposed to be filled with a fluid (cerebrospinal fluid). The aim of this work is to investigate the dynamic characteristics of the human head. The major assumptions of the system considered are that the fluid is inviscid and irrotational and the skull bones and brain tissue are linear, elastic, isotropic and homogeneous. The analysis is based on the three dimensional theory of elasticity and the representation of the displacement field of the skull and brain in terms of the Navier eigenvectors [12]. The motion of the fluid is supposed to undergo small oscillations governed by the wave equation. The characteristic (frequency) equation of the problem under discussion was solved numerically and results for the eigenfrequencies and mode shapes are presented in tables and graphs. The results obtained are compared with experimental ones. As it is expected the results of Engin and Liu are in better agreement with our results as the shell becomes thinner. The results of Guarino and Elger are in quantitative disagreement with our results and it is supposed to be due to the assumptions introduced in Ref. 8.

From the present analysis we see that the three dimensional analysis predicts an extra pattern of natural frequencies on the frequency spectra of the system considered. We note that the cases of fluid-filled hollow sphere, of three elastic spheres and that of an elastic sphere are also discussed and the role of various parameters on the natural frequencies of the system considered are extensively discussed.

2. PROBLEM FORMULATION

The selected model (Fig. 1) consists of an elastic sphere (2 - skull) containing a concentrically located elastic sphere (0 - brain). The space between the regions 2 and 0 is considered to be filled with a fluid (1 - CSF: cerebrospinal fluid)

![Problem Geometry](image)

The skull bones and brain tissue are assumed to be linear, elastic, isotropic and homogeneous while the fluid is considered inviscid and irrotational. The spherical co-ordinate system is shown in Fig. 2.
For motions in homogeneous isotropic, elastic solids the displacement vector field \( u = u(r, t) \), satisfies the equation

\[
\mu_k \nabla^2 u^{(k)}(r, t) + (\lambda_k + \mu_k) \nabla \nabla \cdot u^{(k)}(r, t) = \rho_k \frac{\partial^2 u^{(k)}(r, t)}{\partial t^2},
\]

where \( \mu_k, \lambda_k \) are Lame's constants, \( \rho_k \) is the mass density, \( \nabla \) is the usual del operator, \( t \) is the time and \( k=0,2 \) indicates the brain (\( k=0 \)) and the skull (\( k=2 \)), respectively. We note that \( \lambda_k + \mu_k = K_k + \frac{\mu_k}{3} \), where \( K_k \) is the bulk modulus of the \( k \)-medium.

For the free vibration problem we assume that

\[
u^{(k)}(r, t) = R(t) e^{\omega t},
\]

where \( \omega \) denotes the angular frequency measured in radians/sec and \( i = \sqrt{-1} \).

Introducing the dimensionless variables

\[
r' = \frac{r}{\alpha}, \quad \Omega = \frac{\omega \alpha}{c_{p_2}}, \quad (\alpha = r_2)
\]

and (2) into the equation of motion (1) we obtain

\[
c_{sk}^2 \nabla^2 u^{(k)}(r') + (c_{sk}^2 - c_{sk}^2) \nabla (\nabla \cdot u^{(k)}(r')) + \Omega^2 u^{(k)}(r') = 0,
\]

where

\[
\nabla' = \alpha \nabla, \quad c_{sk}' = \frac{c_{sk}}{c_{p_2}}, \quad c_{sk}' = \frac{c_{sk}}{c_{p_2}}, \quad c_{sk}^2 = \frac{k}{\rho_k}, \quad c_{sk}^2 = \frac{(\lambda_k + 2\mu_k)}{\rho_k}, \quad k = 0,2.
\]
The motion of an inviscid and irrotational fluid undergoing small oscillations is governed by the wave equation,

$$\nabla^2 \Phi = \frac{1}{c_f^2} \frac{\partial^2 \Phi}{\partial t^2}$$

or

$$\nabla^2 \Phi = \frac{1}{c_f^2} \frac{\partial^2 \Phi}{\partial t^2}.$$  \hspace{1cm} (5)

where \( \Phi \) is the velocity potential, \( c_f \) is the speed of sound in the fluid and

$$t = \frac{\alpha}{c_{p_2}} t', \quad c_f' = \frac{c_f}{c_{p_2}}.$$

Supposing that

$$\Phi(r', t') = \text{Re}[\Phi(r', e^{-i\Omega t'})]$$

we obtain

$$\nabla^2 \Phi_{\alpha}(r') = -k_{f}^{12} \Phi_{\alpha}(r')$$  \hspace{1cm} (6)

where \( k_{f} = \Omega/c_{f}' \).

The pressure, \( P \), in the fluid is found from the velocity potential as

$$P = -\rho_f \left( \frac{\partial \Phi}{\partial t} \right).$$  \hspace{1cm} (7)

where \( \rho_f \) is the mass density of the fluid.

The previous relation in dimensionless form is given as

$$P' = \frac{P}{\mu_2} = i\Omega \rho_f' \left( \frac{1}{c_{2s}} \right) \Phi_{\alpha},$$  \hspace{1cm} (8)

where \( \rho_f' = \rho_f/\rho_2 \).

The interaction of the \( k \) media of the system enters the mathematical formulation through the boundary conditions on the surfaces, \( S_k, \ k = 0, 1, 2 \).

We assume that the surface \( S_2 \) is stress-free, that is

$$T_2 u^{(2)}(r_2') = 0$$  \hspace{1cm} (9)

The boundary conditions on \( S_k, k = 1, 0 \) are given as

$$T_2 u^{(2)}(r_1') = -P(r_1') \hat{r}$$  \hspace{1cm} (10a)

$$\hat{r} \cdot u^{(2)}(r_1') = \hat{r} \cdot u^{(1)}(r_1')$$  \hspace{1cm} (10b)

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and
\[ T_0 \mathbf{u}^{(0)}(r_0') = -P(r_0') \hat{r} \]  \hspace{1cm} (11a)
\[ \hat{r} \cdot \mathbf{u}^{(0)}(r_0') = \hat{r} \cdot \mathbf{u}_1^{(0)}(r_0'), \]  \hspace{1cm} (11b)
respectively,

where
\[ T_i = 2 \mu_i' \hat{r} \cdot \nabla + \lambda_i' \hat{r} \times \nabla \cdot + \mu_i' \hat{r} \times \nabla, \quad i = 0, 2 \]  \hspace{1cm} (12)
stands for the dimensionless surface traction operator in \( V_i \), \( \hat{r} \) is the unit outward normal vector on \( S_i \), and \( (\lambda_i', \mu_i') = (\lambda_i/\mu_2, \mu_i/\mu_2) \).

From the physical point of view the conditions (10a, 11a) say that the radial stress in the regions 2 and 0 at \( r = r_0 \) and \( r = r_0' \) is equal and opposite to the fluid pressure at the same surfaces, while (10b, 11b) represent the continuity condition at \( S_k, k = 1, 0 \).

In the course of the present analysis we shall also discuss the cases where
i) the space 1 is filled with an elastic material,
ii) the spaces 0 and 1 are filled with an inviscid and irrotational fluid and
iii) the case of an elastic sphere.

As it is obvious in each case the mathematical problem has to be properly modified. We note that the problem described by the equations of motion (4), (5) and the boundary conditions (9), (10) and (11) is a well-posed mathematical problem.

3. PROBLEM SOLUTION - FREQUENCY EQUATION

In the case of the elastic regions 2 and 0 we represent the displacement field in terms of the Navier eigenvectors [12], that is
\[ \mathbf{u}^{(2)}_0(r) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \sum_{l=1}^{2} \left\{ a^{m,l}_{n,0} L^{m,l}_{n,0}(r) + b^{m,l}_{n,0} M^{m,l}_{n,0}(r) + c^{m,l}_{n,0} N^{m,l}_{n,0}(r) \right\}, \quad r \in V_2 \]  \hspace{1cm} (13)

and
\[ \mathbf{u}^{(0)}_0(r) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left\{ a^{m,1}_{n,0} L^{m,1}_{n,0}(r) + b^{m,1}_{n,0} M^{m,1}_{n,0}(r) + c^{m,1}_{n,0} N^{m,1}_{n,0}(r) \right\}, \quad r \in V_0 \]  \hspace{1cm} (14)
respectively, where
\[ L^{m,l}_{n,0}(r') = \frac{g^{l}_{n}(\Omega r') P^{m}_{n}(\rho)}{\Omega r'} + \sqrt{n(n+1)} \frac{S^{l}_{n}(\Omega r')}{\Omega r'} B^{m}_{n}(\rho) \]
\[ M^{m,l}_{n,0}(r') = \sqrt{n(n+1)} g^{l}_{n}(k_{s} r') C^{m}_{n}(\rho) \]
\[ N^{m,1}_{n,0}(r') = n(n+1) \frac{g^{l}_{n}(k_{s} r')}{k_{s} r'} P^{m}_{n}(\rho) + \sqrt{n(n+1)} \left\{ \frac{g^{l}_{n}(k_{s} r')}{k_{s} r'} + \frac{g^{l}_{n}(k_{s} r')}{k_{s} r'} \right\} B^{m}_{n}(\rho) \]  \hspace{1cm} (15)
\[ \kappa_{p} = \frac{\Omega}{c_{p}}, \quad i = 2 \]
\[ \frac{\Omega}{c_{p}}, \quad i = 0 \]
\[ k_{ij} = \frac{\Omega}{c_{ij}} \begin{cases} \frac{\Omega}{c_{ij}}, & i = 2 \\ \frac{\Omega}{c_{ij}}, & i = 0 \end{cases} \tag{16} \]

\[ \hat{g}^l_i(z) = \frac{d}{dz}(g^l_i), \quad l = 1, 2 \]

and \(g^1_i(z)\) and \(g^2_i(z)\) represent the spherical Bessel functions of the first, \(j_n(z)\), and second, \(y_n(z)\), kind, respectively. The functions \(P^m_n(r')\), \(B^m_n(r')\) and \(C^m_n(r')\) defined on the unit sphere, are the vector spherical harmonics introduced by Hansen and in spherical polar coordinates \((r, \theta, \varphi)\) are given as follows

\[ P^m_n(\hat{r}) = \hat{r} Y^m_n(\hat{r}) \]

\[ B^m_n(\hat{r}) = \frac{1}{\sqrt{n(n+1)}} \left\{ \hat{\theta} \frac{\partial}{\partial \theta} + \hat{\varphi} \frac{1}{\sin \theta} \frac{\partial}{\partial \varphi} \right\} Y^m_n(\hat{r}) \]

\[ C^m_n(\hat{r}) = \frac{1}{\sqrt{n(n+1)}} \left\{ \hat{\theta} \frac{1}{\sin \theta} \frac{\partial}{\partial \varphi} - \hat{\varphi} \frac{\partial}{\partial \theta} \right\} Y^m_n(\hat{r}), \tag{17} \]

where \(\hat{\theta}\) and \(\hat{\varphi}\) are the unit vectors in \(\theta\) and \(\varphi\) directions, respectively, \(Y^m_n(\hat{r}) = P^m_n(\cos \theta)e^{im\varphi}\) are the spherical harmonics and \(P^m_n(\cos \theta)\) are the well known Legendre functions.

The spherical polar coordinates of the real part of the displacement field as well as the expressions \(T_i L^m_{nli} (\hat{r})\), \(T_i M^m_{nli} (\hat{r})\) and \(T_i N^m_{nli} (\hat{r})\) are given in Appendix A.

In the case of fluid the velocity potential \(\Phi\) is given as

\[ \Phi_o(r') = \sum \sum [c_{nm} g^1_n(k' r') + e d_{mn} g^2_n(k' r')] P^m_n(\cos \theta)e^{imp} \tag{18} \]

where

\[ \varepsilon = \begin{cases} 1, & r_0 \neq 0 \\ 0, & r_0 = 0 \end{cases} \]

The velocity of the fluid is expressed as

\[ T_i v^i(r', t') = \nabla \Phi = k' \sum \sum [c_{nm} g^1_n(k' r') + e d_{mn} g^2_n(k' r')] P^m_n(\hat{r}) \]

\[ + \frac{1}{\sqrt{n(n+1)}} B^m_n(\hat{r}) e^{-2\varepsilon N^m_{nli}(\hat{r})} \]

and the corresponding displacement field is

\[ u^{(1)}(r', t') = \frac{i}{\Omega} v(r') = \sum \sum \sum [c_{nm} g^1_n(k' r') + e d_{mn} g^2_n(k' r')] P^m_n(\hat{r}) \]

\[ + \frac{1}{\sqrt{n(n+1)}} B^m_n(\hat{r}) e^{-i2\varepsilon} \tag{19} \]
The expression for the pressure is found by replacing the expression (18) for the velocity potential \( \Phi \) into (8), that is

\[
P = i \Omega \rho' \frac{1}{c^2_s} \Phi = i \Omega \rho' \frac{1}{c^2_s} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left[ \frac{c_n s_n^l (k' r')}{k' r'} + \epsilon d_{mn} s_n^2 (k' r') \right] P_n^m (\cos \theta) e^{im\varphi}. \tag{20}
\]

Since the expressions (13), (14) and (18) satisfy the equations (1) and (5), respectively, it remains to ask the boundary conditions (9), (11) and (12) to be satisfied.

The application of the boundary conditions (for the case \( \nu' \neq 0 \)) give

At \( r = r_2 \):

\[
T_2 \mathbf{u}^{(2)} (r_2) = 0
\]

or

\[
\sum_{l=1}^{2} \left[ \alpha_{n2}^{m,l} A_{n2}^l (r_2') + \gamma_{n2}^{m,l} D_{n2}^l (r_2') \right] = 0 \tag{21a}
\]

\[
\sum_{l=1}^{2} \left[ \alpha_{n2}^{m,l} B_{n2}^l (r_2') + \gamma_{n2}^{m,l} E_{n2}^l (r_2') \right] = 0 \tag{21b}
\]

\[
\sum_{l=1}^{2} \beta_{n2}^{m,l} \Gamma_{n2}^l (r_2') = 0. \tag{21c}
\]

At \( r = r_1 \):

\[
T_2 \mathbf{u}^{(2)} (r_1) = -P (r_1) \hat{r}
\]

or

\[
\sum_{l=1}^{2} \left[ \alpha_{n2}^{m,l} A_{n2}^l (r_1') + \gamma_{n2}^{m,l} D_{n2}^l (r_1') \right] = -i \Omega \rho' \frac{1}{c^2_s} \sum_{l=1}^{2} \left[ c_{mn} s_n^l (k' r_1') + d_{mn} s_n^2 (k' r_1') \right] \tag{22a}
\]

\[
\sum_{l=1}^{2} \left[ \alpha_{n2}^{m,l} B_{n2}^l (r_1') + \gamma_{n2}^{m,l} E_{n2}^l (r_1') \right] = 0 \tag{22b}
\]

\[
\sum_{l=1}^{2} \beta_{n2}^{m,l} \Gamma_{n2}^l (r_1') = 0 \tag{22c}
\]

and

\[
\hat{r} \cdot \mathbf{u}^{(2)} (r_1') = \hat{r} \cdot \mathbf{u}^{(1)} (r_1');
\]

or
\[
\sum_{l=0}^{n} \left[ \alpha_{n,0}^{m,l} \cdot g_n^l \left( \Omega r'_1 \right) + \gamma_{n,0}^{m,l} n(n+1) \frac{g_n^l(k'_s r'_1)}{k'_s r'_1} \right] = \]
\[
\frac{i}{c f} \left[ c_{nm} \tilde{g}_n^l(k'_f r'_0) + d_{nm} \tilde{g}_n^l(k'_f r'_0) \right]
\]

At \( r = r'_0 \):

\[
T_{r'_0} u^{(0)}(r'_0) = -P(r'_0) \hat{r}
\]

or

\[
\alpha_{n,0}^{m,l} A_{n,0}^l(r'_0) + \gamma_{n,0}^{m,l} D_{n,0}^l(r'_0) =
\]
\[
-\Omega \rho_f \frac{1}{c f} \cdot \left[ c_{nm} \tilde{g}_n^l(k'_f r'_0) + d_{nm} \tilde{g}_n^l(k'_f r'_0) \right]
\]

\[
\alpha_{n,0}^{m,l} B_{n,0}^l(r'_0) + \gamma_{n,0}^{m,l} E_{n,0}^l(r'_0) = 0
\]

\[
\beta_{n,0}^{m,l} \Gamma_{n,0}^l(r'_0) = 0
\]

and

\[
\hat{r} \cdot u^{(1)}(r'_0) = \hat{r} \cdot u^{(0)}(r'_0)
\]

or

\[
\alpha_{n,0}^{m,l} \cdot g_n^l \left( \Omega r'_0 \right) + \gamma_{n,0}^{m,l} n(n+1) \frac{g_n^l(k'_s r'_0)}{k'_s r'_0} = \]
\[
\frac{i}{c f} \left[ c_{nm} \tilde{g}_n^l(k'_f r'_0) + d_{nm} \tilde{g}_n^l(k'_f r'_0) \right]
\]

where

\[
A_{n,i}^l(r') = \left[ \frac{4 \mu_{l'}}{r'} g_n^l(k'_{pi} r') + 2 \mu_{l'} k'_{pi} \left( 1 - \frac{n(n+1)}{k'_{pi}^2 r'^2} \right) g_n^l(k'_{pi} r') + \lambda_{l'} k'_s g_n^l(k'_{pi} r') \right]
\]

\[
B_{n,i}^l(r') = 2 \mu_{l'} \sqrt{n(n+1)} \left[ \frac{1}{r'} \tilde{g}_n^l(k'_{pi} r') - \frac{g_n^l(k'_{pi} r')}{k'_{pi} r'^2} \right]
\]

\[
\Gamma_{n,i}^l(r') = \mu_{l'} \sqrt{n(n+1)} \left[ k'_s \tilde{g}_n^l(k'_s r') - \frac{1}{r'} \tilde{g}_n^l(k'_s r') \right]
\]

\[
D_{n,i}^l(r') = 2 \mu_{l'} \cdot n(n+1) \left[ \frac{\tilde{g}_n^l(k'_s r')}{r'} - \frac{g_n^l(k'_s r')}{k'_s r'^2} \right]
\]
\[ E_{x_i}(r') = \mu_i \sqrt{n(n+1)} \left[ -2 \frac{g_s^i(k_{si} r')}{r'} + k_{si} g_s^i(k_{si} r') + 2 \frac{n(n+1)-1}{k_{si} r'^2} g_s^i(k_{si} r') \right]. \]

The system of the algebraic equations (21) - (25) can be written as

\[ D\mathbf{x} = 0. \tag{26} \]

Details about the system (26), for each case under discussion, are given in Appendix B.

In order for the system (26) to have a non-trivial solution, the following condition has to be satisfied

\[ \text{det}(D) = 0. \tag{27} \]

This condition provides the characteristic (frequency) equation, the roots of which are the eigenfrequency coefficients \( \Omega \) of the system under discussion. The mode shape corresponding to each \( \Omega \) is obtained by solving the system (26).

4. NUMERICAL RESULTS - DISCUSSION

The frequency equation (27) has been solved numerically and for this purpose a matrix determinant computation routine was used for different frequency coefficients \( \Omega \), along with a bisection method to refine steps close to its roots. The root finding algorithm is followed by an LU-decomposition and back-substitution routine to determine the eigenvectors, whose elements are used for the computation of the corresponding displacement components.

The main purpose of our analysis is the modelling of the cranial system. The system is composed from the skull bones, the cerebrospinal fluid and the brain. In the present analysis we assume that the properties of the cranial components are:

- **Skull** [13]
  \[ E = 6.5 \times 10^9 \text{ N/m}^2, \quad \nu = 0.25, \quad \rho = 2.1326 \times 10^3 \text{ Kg/m}^3. \]

- **Cerebrospinal Fluid (CSF)** [14]
  \[ K = 2.1029753 \times 10^9 \text{ N/m}^2, \quad \rho = 1.0002 \times 10^3 \text{ Kg/m}^3. \]

- **Brain** [15]
  \[ K = 2.102975 \times 10^9 \text{ N/m}^2, \quad \rho = 1.007 \times 10^3 \text{ Kg/m}^3. \]

Fluid: \( K = 2.102975 \times 10^9 \text{ N/m}^2, \quad \rho = 1.007 \times 10^3 \text{ Kg/m}^3 \)

Elastic: \( E = 2.522357 \times 10^8 \text{ N/m}^2, \quad \nu = 0.48, \quad \rho = 1.007 \times 10^3 \text{ Kg/m}^3 \)

where \( E, K \) and \( \nu \) denote the Young modulus, bulk modulus and Poisson ratio, respectively. The elastic material constants are related as \( \mu = \frac{E}{2(1+\nu)} \) and \( \lambda = \frac{E \nu}{(1+\nu)(1-2\nu)} \),

and the geometry of the system is defined by

\[ r_2 = 0.082 \text{ m}, \quad r_1 = 0.076 \text{ m}, \quad r_0 = 0.070 \text{ m}. \]

In Ref. 12 the authors of the present work presented the dynamic characteristics of the human skull which are in good agreement with the existing experimental ones. For the purpose of comparison, in Table 1 the reproduced frequency coefficients of Ref. 4 are shown, as well as those obtained by the present analysis. As it is expected, the results of Engin and Liu are in better agreement with our results (three dimensional theory of elasticity), as the shell becomes thinner. In the case of an elastic spherical shell in vacuo (skull) we observe that for \( h/R=20 \) the difference between the compared results is about 3.1 % while for \( h/R=10 \) the difference becomes 7.5 %.
Table 1: Eigenfrequency Coefficients $\tilde{\Omega}_n = \omega \alpha/c_r$ of Engin and Liu [4]

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Fluid Filled Shell (v=0.3, l=9.38, s=0.553, h/R=1/20)

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<td>Engin &amp; Liu</td>
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<td>5.5151</td>
<td>4.867</td>
<td>6.6986</td>
<td>8.3155</td>
</tr>
</tbody>
</table>

Skull in vacuo (v=0.3, l=0, s=0.553, h/R=1/20)

<table>
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<tr>
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<td>Engin &amp; Liu</td>
<td>Present Analysis</td>
<td>Engin &amp; Liu</td>
<td>Present Analysis</td>
<td>Engin &amp; Liu</td>
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<td>3.3007</td>
<td>1.2801</td>
<td>1.2151</td>
<td>1.5784</td>
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<tr>
<td>n</td>
<td>Fluid Filled Rigid shell</td>
<td>Fluid filled elastic shell with 2 cm rigid sphere</td>
<td>Fluid filled elastic shell with 1 cm elastic sphere</td>
<td>Elastic shell in vacuo</td>
<td>Fluid filled elastic shell</td>
<td>Fluid filled elastic shell with 2 cm stiff sphere</td>
<td>Fluid filled elastic shell with 2 cm elastic sphere</td>
</tr>
<tr>
<td>----</td>
<td>--------------------------</td>
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</tr>
<tr>
<td>0</td>
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<td>4.7272</td>
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<td>5.9259</td>
<td>3.0976</td>
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<td>3.1733</td>
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<td>5.4971</td>
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<tr>
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<td>3.0184</td>
<td>3.6056</td>
<td>5.7670</td>
<td>5.7670</td>
<td>0.7157</td>
<td>0.8261</td>
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<tr>
<td></td>
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<td>7.6083</td>
<td>9.3930</td>
<td>9.3930</td>
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<td>0.8267</td>
</tr>
<tr>
<td>3</td>
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<td>4.7349</td>
<td>5.9259</td>
<td>5.9259</td>
<td>0.0921</td>
<td>1.0459</td>
<td>1.0459</td>
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<tr>
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<td>8.2063</td>
<td>9.1039</td>
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<td>10.1301</td>
<td>0.0921</td>
<td>1.0704</td>
<td>1.0704</td>
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<td>4</td>
<td>5.4201</td>
<td>6.7321</td>
<td>8.2158</td>
<td>8.2158</td>
<td>1.2638</td>
<td>1.3265</td>
<td>1.3265</td>
</tr>
</tbody>
</table>
From the results obtained we see that for \( n \geq 2 \) the three-dimensional theory predicts an extra pattern of natural frequencies on the frequency spectra of the system considered (fluid-filled and in vacuo elastic sphere).

The fluid-filled elastic shell containing an elastic sphere was studied in Ref. 8 where extensive results for the natural frequencies are cited. The results of Ref. 8 as well as those using the present analysis are presented in Table 2, with material constants [8]:

\[
E_2 = 1.38 \times 10^6 \frac{N}{m^2}, \quad \nu_2 = 0.25, \quad \rho_2 = 2132.6 \frac{Kg}{m^3}, \quad r_2 = 0.08001m
\]
\[
c_1 = 1450 \frac{m}{s}, \quad \rho_1 = 1000 \frac{Kg}{m^3}, \quad r_1 = 0.07239m
\]
\[
E_0 = 3.4 \times 10^8 \frac{N}{m^2}, \quad \nu_0 = 0.478, \quad \rho_0 = 1070 \frac{Kg}{m^3} \text{(elastic sphere)}
\]
\[
E_0 = 1 \times 10^{10} \frac{N}{m^2}, \quad \nu_0 = 0.25, \quad \rho_0 = 1000 \frac{Kg}{m^3} \text{(stiff sphere)}.
\]

From the results comparison one can lead to the conclusion that there is a great quantitative disagreement between our results and those of Ref. 8 even in the case of skull in vacuo, where the present results are in excellent agreement with the previous ones existing in the literature [2]. The reason of the disagreement observed, perhaps, is due to the assumptions adopted by Guarino and Elger.

In Table 3 the first ten eigenfrequency coefficients, \( \Omega^{(k)}, k = 1, 2, \ldots, 10 \), are cited for the models considered (skull - S, fluid-filled skull - FF, fluid-filled skull with an elastic core - FFE, three elastic spheres - EEE, and elastic sphere - E) by using the present analysis. We observe that for the \( k \)-th eigenfrequency of each model we have the arrangement

\[
\Omega^{(k)}(FFE) < \Omega^{(k)}(FF) < \Omega^{(k)}(EEE) < \Omega^{(k)}(S) < \Omega^{(k)}(E)
\]

where \( k = 1, 2, \ldots, 8 \). This arrangement for \( k > 8 \) ceases to exist.

Table 3: Comparison of \( \Omega^{(k)}, k = 1, 2, \ldots, 10 \)

<table>
<thead>
<tr>
<th></th>
<th>Fluid Filled with an Elastic Core (FFE)</th>
<th>Fluid Filled Skull (FF)</th>
<th>Three Elastic Spheres (EEE)</th>
<th>Skull (S)</th>
<th>Elastic Sphere (E)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.2109 (n=2)</td>
<td>0.4082 (n=2)</td>
<td>0.5709 (n=2)</td>
<td>0.7863 (n=2)</td>
<td>1.0038 (n=0)</td>
</tr>
<tr>
<td>2</td>
<td>0.3252 (n=3)</td>
<td>0.5285 (n=3)</td>
<td>0.7308 (n=3)</td>
<td>0.8522 (n=3)</td>
<td>1.0211 (n=2)</td>
</tr>
<tr>
<td>3</td>
<td>0.4378 (n=4)</td>
<td>0.6291 (n=4)</td>
<td>0.8709 (n=4)</td>
<td>0.9486 (n=4)</td>
<td>1.0847 (n=2)</td>
</tr>
<tr>
<td>4</td>
<td>0.5549 (n=5)</td>
<td>0.7437 (n=5)</td>
<td>0.9653 (n=5)</td>
<td>1.0631 (n=5)</td>
<td>1.5174 (n=1)</td>
</tr>
<tr>
<td>5</td>
<td>0.7006 (n=6)</td>
<td>0.8556 (n=6)</td>
<td>1.0577 (n=6)</td>
<td>1.1971 (n=6)</td>
<td>1.5798 (n=3)</td>
</tr>
<tr>
<td>6</td>
<td>0.9029 (n=2)</td>
<td>1.0601 (n=7)</td>
<td>1.2409 (n=3)</td>
<td>1.2184 (n=6)</td>
<td>1.6220 (n=3)</td>
</tr>
<tr>
<td>7</td>
<td>0.9514 (n=1)</td>
<td>1.1971 (n=2)</td>
<td>1.3019 (n=4)</td>
<td>1.4210 (n=7)</td>
<td>1.7452 (n=0)</td>
</tr>
<tr>
<td>8</td>
<td>0.9677 (n=2)</td>
<td>1.2092 (n=1)</td>
<td>1.3209 (n=0)</td>
<td>1.5479 (n=1)</td>
<td>2.0799 (n=4)</td>
</tr>
<tr>
<td>9</td>
<td>1.1065 (n=7)</td>
<td>1.3405 (n=8)</td>
<td>1.3279 (n=1)</td>
<td>1.6695 (n=8)</td>
<td>2.1481 (n=2)</td>
</tr>
<tr>
<td>10</td>
<td>1.1971 (n=2)</td>
<td>1.8928 (n=3)</td>
<td>1.5505 (n=4)</td>
<td>1.8928 (n=3)</td>
<td>2.3529 (n=1)</td>
</tr>
</tbody>
</table>
As it is obvious the eigenfrequencies of the system depend on the geometry of the system as well as on its physical properties. For the FF-model we parametrized the first eigenfrequency coefficients $\Omega_n$ as

$$\min \Omega_n = \min \overline{\Omega}_n(\nu, \frac{\rho_1}{\rho_2}, \frac{r_2 - r_1}{r_2}), \quad n = 0,1,\ldots,4$$

and the results are cited in Table 4. We observe that as $\frac{\rho_1}{\rho_2}$ is decreasing $\min \Omega_n$ is increasing and as $\nu$ is increasing $\min \Omega_n$ is decreasing (for the same $n$).

Table 4: FF-model: Parametrization of $\min \Omega_n$, $n = 0,1,\ldots,4$

<table>
<thead>
<tr>
<th>variation of $\frac{r_2 - r_1}{r_2}$</th>
<th>$n=0$</th>
<th>$n=1$</th>
<th>$n=2$</th>
<th>$n=3$</th>
<th>$n=4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thin Shell</td>
<td>5.2579</td>
<td>1.0544</td>
<td>1.1548</td>
<td>1.8259</td>
<td>2.4496</td>
</tr>
<tr>
<td>0.146</td>
<td>4.3272</td>
<td>1.3516</td>
<td>0.5497</td>
<td>0.7362</td>
<td>0.9470</td>
</tr>
<tr>
<td>0.268</td>
<td>3.6429</td>
<td>1.5724</td>
<td>0.7278</td>
<td>1.0725</td>
<td>1.5076</td>
</tr>
<tr>
<td>0.390</td>
<td>3.1801</td>
<td>1.7726</td>
<td>0.9014</td>
<td>1.4277</td>
<td>2.0044</td>
</tr>
<tr>
<td>0.634</td>
<td>2.7043</td>
<td>1.9955</td>
<td>1.2736</td>
<td>2.0755</td>
<td>2.8057</td>
</tr>
<tr>
<td>0.817</td>
<td>2.5827</td>
<td>2.1768</td>
<td>1.4436</td>
<td>2.2312</td>
<td>2.8912</td>
</tr>
<tr>
<td>Elastic Sphere</td>
<td>2.5634</td>
<td>1.9772</td>
<td>1.4440</td>
<td>2.2313</td>
<td>2.8921</td>
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</table>

<table>
<thead>
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<th>$n=3$</th>
<th>$n=4$</th>
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<td>1.407</td>
<td>3.1327</td>
<td>1.1104</td>
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<td>0.3534</td>
<td>0.4316</td>
</tr>
<tr>
<td>1.0</td>
<td>3.6265</td>
<td>1.1329</td>
<td>0.3074</td>
<td>0.4053</td>
<td>0.4910</td>
</tr>
<tr>
<td>0.703</td>
<td>4.1736</td>
<td>1.1634</td>
<td>0.3531</td>
<td>0.4619</td>
<td>0.5558</td>
</tr>
<tr>
<td>0.469</td>
<td>4.8187</td>
<td>1.2092</td>
<td>0.4083</td>
<td>0.5285</td>
<td>0.6291</td>
</tr>
<tr>
<td>0.234</td>
<td>7.1520</td>
<td>1.7401</td>
<td>0.6760</td>
<td>0.8204</td>
<td>0.9199</td>
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</table>

<table>
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<td>4.9898</td>
<td>1.2241</td>
<td>0.4277</td>
<td>0.5517</td>
<td>0.6551</td>
</tr>
<tr>
<td>0.25</td>
<td>4.8187</td>
<td>1.2082</td>
<td>0.4083</td>
<td>0.5285</td>
<td>0.6291</td>
</tr>
<tr>
<td>0.30</td>
<td>4.5690</td>
<td>1.1737</td>
<td>0.3825</td>
<td>0.4970</td>
<td>0.5932</td>
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<tr>
<td>0.40</td>
<td>3.6584</td>
<td>0.9881</td>
<td>0.2985</td>
<td>0.3909</td>
<td>0.4695</td>
</tr>
<tr>
<td>0.48</td>
<td>1.8258</td>
<td>0.5150</td>
<td>0.1457</td>
<td>0.1920</td>
<td>0.2321</td>
</tr>
</tbody>
</table>

For the model (FFE) we studied the parametrization

$$\min \Omega_n = \min \overline{\Omega}_n(\nu_i, \nu_0, \frac{r_2 - r_1}{r_2}, \frac{r_0}{r_2}), \quad n = 0,1,\ldots,4$$

and the results are presented in Table 5. It is observed that the effect of $\nu_i, i = 0,2$ on $\min \Omega_n$ is analogous to that we discussed for the (FF) model, while the effect of increment of $\frac{r_2 - r_1}{r_2}$ or $\frac{r_0}{r_2}$ is increment of $\min \Omega_n$, $n = 0,1,\ldots,4$ (for same $n$).
Table 5: FFE - model: parametrization of $\min \Omega_n$, $n = 0, 1, \ldots, 4$

<table>
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<tr>
<th>$\frac{r_0}{r_2}$</th>
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<td>No Fluid</td>
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<td>1.2573</td>
<td>1.5676</td>
</tr>
<tr>
<td>0.854</td>
<td>0.9514</td>
<td>1.2079</td>
<td>0.2109</td>
<td>0.3252</td>
<td>0.4378</td>
</tr>
<tr>
<td>0.732</td>
<td>1.1099</td>
<td>1.2083</td>
<td>0.3232</td>
<td>0.4648</td>
<td>0.5659</td>
</tr>
<tr>
<td>0.610</td>
<td>1.3319</td>
<td>1.2085</td>
<td>0.3758</td>
<td>0.5116</td>
<td>0.6212</td>
</tr>
<tr>
<td>0.366</td>
<td>2.2199</td>
<td>1.2066</td>
<td>0.4050</td>
<td>0.5281</td>
<td>0.6291</td>
</tr>
<tr>
<td>0.183</td>
<td>4.4358</td>
<td>1.2066</td>
<td>0.4082</td>
<td>0.4457</td>
<td>0.6291</td>
</tr>
<tr>
<td>Fluid Filled</td>
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<td>1.2092</td>
<td>0.4083</td>
<td>0.5285</td>
<td>0.6291</td>
</tr>
<tr>
<td>Elastic Sphere</td>
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</table>

<table>
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<td>1.2228</td>
<td>0.2204</td>
<td>0.3390</td>
<td>0.4556</td>
</tr>
<tr>
<td>0.25</td>
<td>0.9514</td>
<td>1.2079</td>
<td>0.2109</td>
<td>0.3252</td>
<td>0.4378</td>
</tr>
<tr>
<td>0.30</td>
<td>0.8982</td>
<td>1.1724</td>
<td>0.1960</td>
<td>0.3061</td>
<td>0.4130</td>
</tr>
<tr>
<td>0.40</td>
<td>0.7119</td>
<td>0.9666</td>
<td>0.1553</td>
<td>0.2413</td>
<td>0.3272</td>
</tr>
<tr>
<td>0.48</td>
<td>0.3516</td>
<td>0.5139</td>
<td>0.0761</td>
<td>0.1188</td>
<td>0.1538</td>
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<table>
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<th>$n=0$</th>
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<th>$n=3$</th>
<th>$n=4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.20</td>
<td>4.0920</td>
<td>1.2080</td>
<td>0.2266</td>
<td>0.3406</td>
<td>0.4525</td>
</tr>
<tr>
<td>0.25</td>
<td>3.6600</td>
<td>1.2080</td>
<td>0.2264</td>
<td>0.3404</td>
<td>0.4523</td>
</tr>
<tr>
<td>0.30</td>
<td>3.2101</td>
<td>1.2080</td>
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<tr>
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<td>0.2243</td>
<td>0.3384</td>
<td>0.4504</td>
</tr>
<tr>
<td>0.48</td>
<td>0.9514</td>
<td>1.2079</td>
<td>0.2109</td>
<td>0.3252</td>
<td>0.4378</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\frac{r_2 - r_1}{r_2}$</th>
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<th>$n=1$</th>
<th>$n=2$</th>
<th>$n=3$</th>
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</tr>
</thead>
<tbody>
<tr>
<td>0.073</td>
<td>0.9514</td>
<td>1.2079</td>
<td>0.2109</td>
<td>0.3252</td>
<td>0.4378</td>
</tr>
<tr>
<td>0.146</td>
<td>1.0406</td>
<td>1.3494</td>
<td>0.3126</td>
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<td>0.7220</td>
</tr>
<tr>
<td>0.268</td>
<td>1.2043</td>
<td>1.5695</td>
<td>0.4323</td>
<td>0.7503</td>
<td>1.1438</td>
</tr>
<tr>
<td>0.390</td>
<td>1.4446</td>
<td>1.7701</td>
<td>0.5780</td>
<td>1.0306</td>
<td>1.5489</td>
</tr>
<tr>
<td>0.634</td>
<td>2.4129</td>
<td>1.9949</td>
<td>1.0528</td>
<td>1.8527</td>
<td>2.6809</td>
</tr>
</tbody>
</table>

In Figure 3 are shown the frequency spectra for the models considered in the present study. From these results one can see the influence of the cerebrospinal fluid and the brain on the natural frequencies of the human skull. The inner elastic sphere (brain) introduces a pattern of additional frequencies to the spectra of eigenfrequencies of the physical system. The same happens with the EEE - model.

Mode shapes for selected eigenfrequencies are presented in Fig. 4. The total mode of the physical system corresponding to $\Omega_n$ is the superposition of the displacements components $(u_r, u_\theta, u_z)$. The general shape of $u_r, u_\theta, u_z$ depend upon the order of $n$ while the relative amplitudes of them corresponding to a particular frequency $\Omega_n$ depend upon the value of $\Omega_n$.
Figure 3: Eigenfrequency spectra of the E-, S-, FF-, FFE- and EEE- models.
Figure 4: FFE - Model: Mode shapes for \((u_r, u_\theta, u_\phi)\)

(A: \(n=1\ \Omega_r=2.0247\), B: \(n=2\ \Omega_r=0.8259\), C: \(n=3\ \Omega_r=1.0793\), D: \(n=4\ \Omega_r=1.3145\))
Finally the results obtained are compared with the experimental ones presented in Ref. 3. Following the approximation adopted by Engin and Liu [4] and many other researchers about the cranial system geometry and its physical properties we used the present analysis to calculate the eigenfrequencies of the FF - and FFE - models. For the purpose of comparison the results obtained as well as the experimental ones are presented in Table 6 and graphically in Fig. 5.

We observe that:
\[ \omega_{\text{FFE}}^{(k)} < \omega_{\text{exp}}^{(k)} < \omega_{\text{FF}}^{(k)}, \quad k = 1, 2, \ldots, 19 \]

and also that the resonance frequencies are > 483 Hz. The analysis on the basis of the FFE - and FF - models predicts nineteen and six frequencies ≤ 3650 Hz, respectively, while the measured ones are only ten.

**Table 6:** Eigenfrequencies (Hz) of the human skull - brain system [3].

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
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It is remarkable to note that the estimation of the resonance frequencies of the human skull in vivo by the mean value of the corresponding ones to the FF - and FFE - models is in the range of the measurements presented by Hékansson et al., although we did not take into account the neck support as well as the viscous damping of the physical system.
5. CONCLUDING REMARKS

In this work we presented a mathematical analysis for the study of the dynamic characteristics of the human skull-brain system in the framework of the three-dimensional theory of elasticity. The geometry of the physical system considered has been modeled by the FFE-model where the outer hollow elastic sphere represents the skull, the inner elastic sphere corresponds to the brain, and the space between them is supposed to be filled with the cerebrospinal fluid. The related FF-, EEE-, S-, and E-models are also discussed.

The proposed analysis was used to calculate the eigenfrequencies and the corresponding to them eigenmodes of the simulated physical system. The results obtained are in good agreement with the existing theoretical ones and further study is needed to simulate the real physical system. The reason of the disagreement between our results and the experimental ones are supposed to be due to the approximation of the materials entering the problem as well as the absence of the influence on the dynamic characteristics of the neck support. We observed that the presence of the inner sphere changes the natural frequencies spectra of a fluid-filled hollow elastic sphere. This change increases when the shell theory is extended to the three-dimensional theory of elasticity.

The dynamic analysis of the head-neck system as well as the simulation of the brain matter by a viscoelastic material are in preparation and will be presented in the future.
ACKNOWLEDGEMENT

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REFERENCES


APPENDIX A:

The spherical polar coordinates of the real part of the displacement field for the region 2 are:

\[
\begin{align*}
\mathbf{u}_{or}^{(2)} &= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \sum_{l=0}^{2} \left\{ \alpha_{n,2}^{m,l} g_n^l(k^\prime s, r') \cos(m\phi) P_n^m(\cos \theta) + \right. \\
&\left. \gamma_{n,2}^{m,l} (n(n+1)) \frac{g_n^l(k^\prime s, r')}{k^\prime s, r'} \cos(m\phi) P_n^m(\cos \theta) \right\} \\
\mathbf{u}_{ob}^{(2)} &= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \sum_{l=0}^{2} \left\{ \alpha_{n,2}^{m,l} \frac{g_n^l(\Omega r')}{\Omega r'} \cos(m\phi) \frac{\partial}{\partial \theta} P_n^m(\cos \theta) \\
&- \beta_{n,2}^{m,l} \frac{g_n^l(k^\prime s, r')}{k^\prime s, r'} \frac{\sin(m\phi)}{\sin \theta} P_n^m(\cos \theta) + \right. \\
&\left. + \gamma_{n,2}^{m,l} \left( g_n^l(k^\prime s, r') + \frac{g_n^l(k^\prime s, r')}{k^\prime s, r'} \cos(m\phi) \right) \frac{\partial}{\partial \theta} P_n^m(\cos \theta) \right\} \\
\mathbf{u}_{op}^{(2)} &= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \sum_{l=0}^{2} \left\{ -\alpha_{n,2}^{m,l} \frac{g_n^l(\Omega r')}{\Omega r'} \frac{\sin(m\phi)}{\sin \theta} P_n^m(\cos \theta) \\
&- \beta_{n,2}^{m,l} g_n^l(k^\prime s, r') \cos(m\phi) \frac{\partial}{\partial \theta} P_n^m(\cos \theta) + \right. \\
&\left. - \gamma_{n,2}^{m,l} \left( g_n^l(k^\prime s, r') + \frac{g_n^l(k^\prime s, r')}{} \right) \frac{\sin(m\phi)}{\sin \theta} P_n^m(\cos \theta) \right\}.
\end{align*}
\]

The expressions \( T_i L_n^{m,l}(\hat{r}) \), \( T_i M_n^{m,l}(\hat{r}) \) and \( T_i N_n^{m,l}(\hat{r}) \) are:

\[
\begin{align*}
T_i L_n^{m,l}(\hat{r}) &= \left[ \frac{4\mu'}{r'} \frac{\hat{g}_n^l(\Omega r')}{\Omega r'} + 2\mu' \Omega (1 - \frac{n(n+1)}{\Omega^2 r'^2}) g_n^l(\Omega r') + \lambda' \Omega g_n^l(\Omega r') \right] P_n^{m}(\hat{r}) \\
&+ 2\mu' \sqrt{n(n+1)} \left[ \frac{\hat{g}_n^l(\Omega r')}{r'} - \frac{\hat{g}_n^l(\Omega r')}{\Omega r'^2} \right] B_n^{m}(\hat{r}) \\
T_i M_n^{m,l}(\hat{r}) &= \mu' \sqrt{n(n+1)} \left[ k^\prime s, g_n^l(k^\prime s, r') - \frac{1}{r'} g_n^l(k^\prime s, r') \right] C_n^{m}(\hat{r}) \\
T_i N_n^{m,l}(\hat{r}) &= 2\mu' n(n+1) \left[ \frac{\hat{g}_n^l(k^\prime s, r')}{r'} - \frac{\hat{g}_n^l(k^\prime s, r')}{k^\prime s, r'^2} \right] P_n^{m}(\hat{r}) \\
&+ \mu' \sqrt{n(n+1)} \left[ -2\frac{\hat{g}_n^l(k^\prime s, r')}{r'} - k^\prime s, g_n^l(k^\prime s, r') + 2\frac{n(n+1)}{k^\prime s, r'^2} g_n^l(k^\prime s, r') \right] B_n^{m}(\hat{r})
\end{align*}
\]
APPENDIX B: \(Dx = 0\), \(D = [d_{ij}]\)

Case 1: Elastic Sphere (E - model)

\[
\begin{bmatrix}
A^1_n(r_2) & 0 & D^1_n(r_2) \\
0 & \Gamma^1_n(r_2) & 0 \\
B^1_n(r_2) & 0 & E^1_n(r_2)
\end{bmatrix}
\begin{bmatrix}
\alpha^n_\alpha \\
\beta^n_\alpha \\
\gamma^n_\alpha
\end{bmatrix} = 0
\]

Case 2: Fluid - Filled Elastic Sphere (FF - model)

\[
\begin{bmatrix}
A^1_n(r_2) & A^2_n(r' \_2) & 0 & 0 & D^1_n(r' \_2) & D^2_n(r' \_2) & 0 & 0 & \alpha^{m,1}_n \\
B^1_n(r_2) & B^2_n(r' \_2) & 0 & 0 & E^1_n(r' \_2) & E^2_n(r' \_2) & 0 & 0 & \alpha^{m,2}_n \\
0 & 0 & \Gamma^1_n(r' \_2) & \Gamma^2_n(r' \_2) & 0 & 0 & 0 & 0 & \beta^{m,1}_n \\
A^1_n(r' \_0) & A^2_n(r' \_0) & 0 & 0 & D^1_n(r' \_0) & D^2_n(r' \_0) & 0 & 0 & \beta^{m,2}_n \\
B^1_n(r' \_0) & B^2_n(r' \_0) & 0 & 0 & E^1_n(r' \_0) & E^2_n(r' \_0) & 0 & 0 & \gamma^{m,1}_n \\
0 & 0 & \Gamma^1_n(r' \_0) & \Gamma^2_n(r' \_0) & 0 & 0 & 0 & 0 & \gamma^{m,2}_n \\
d_{7,1} & d_{7,2} & 0 & 0 & d_{7,5} & d_{7,6} & d_{7,7} & c_{nm}
\end{bmatrix}
\begin{bmatrix}
\alpha^{m,1}_n \\
\alpha^{m,2}_n \\
\beta^{m,1}_n \\
\beta^{m,2}_n \\
\gamma^{m,1}_n \\
\gamma^{m,2}_n
\end{bmatrix} = 0
\]

where

\[
d_{4,3} = -i \Omega \frac{\beta_f}{c_f} g_\alpha(k_{f} r' \_0),\ d_{4,4} = g^{1}_c(\Omega r' \_0),\ d_{4,5} = g^{2}_c(\Omega r' \_0),\ d_{4,6} = n(n+1) \frac{g^{1}_\alpha(k_{f} r' \_0)}{k_{f} r' \_0}
\]

\[
d_{7,6} = n(n+1) \frac{g^{2}_f(k_{f} r' \_0)}{k_{f} r' \_0},\ d_{7,7} = \frac{i}{c_f} \frac{g^{1}_f(k_{f} r' \_0)}{k_{f} r' \_0}
\]
Case 3: Fluid - Filled Elastic Sphere Containing an Elastic Sphere (FFE - model)

In this case the vector $x$ and the non-zero elements of the matrix $D$ are

$$x = \begin{bmatrix} \alpha_{n,1} & \alpha_{n,2}^m & \beta_{n,1} & \beta_{n,2}^m & \gamma_{n,1} & \gamma_{n,2}^m & \alpha_{s,0} & \beta_{s,0}^m & \gamma_{s,0} & \gamma_{s,0}^m & c_m & d_{nn} \end{bmatrix}^T$$

$$d_{1,1} = A_{n,2}^1(r_2'), d_{1,2} = A_{n,2}^2(r_2'), d_{1,5} = D_{n,2}^1(r_2'), d_{1,6} = D_{n,2}^2(r_2'),$$

$$d_{2,1} = E_{n,2}^1(r_2'), d_{2,2} = E_{n,2}^2(r_2'), d_{2,5} = E_{s,2}^1(r_2'), d_{2,6} = E_{s,2}^2(r_2')$$

$$d_{3,3} = \Gamma_{s,2}^1(r_2'), d_{3,4} = \Gamma_{s,2}^2(r_2')$$

$$d_{4,1} = A_{n,1}^1(r_1'), d_{4,2} = A_{n,1}^2(r_1'), d_{4,5} = D_{n,1}^1(r_1'), d_{4,6} = D_{n,1}^2(r_1')$$

$$d_{4,10} = \Omega_{r_1} \frac{1}{c_{r_1}^2} g_n^1(k_f r_1'), d_{4,11} = \Omega_{r_1} \frac{1}{c_{r_1}^2} g_n^2(k_f r_1')$$

$$d_{5,1} = B_{s,2}^1(r_2'), d_{5,2} = B_{s,2}^2(r_2'), d_{5,5} = E_{s,2}^1(r_1'), d_{5,6} = E_{s,2}^2(r_1')$$

$$d_{6,3} = \Gamma_{s,1}^1(r_1'), d_{6,4} = \Gamma_{s,1}^2(r_1')$$

$$d_{7,1} = \frac{g_n^1(k_s, r_1')}{k_{s_2} r_1'}, d_{7,2} = \frac{g_n^2(k_s, r_1')}{k_{s_2} r_1'}, d_{7,5} = \frac{g_s^1(k_f, r_1')}{k_{s_2} r_1'}, d_{7,6} = \frac{g_s^2(k_f, r_1')}{k_{s_2} r_1'}$$

$$d_{7,10} = -\frac{1}{c_f} g_n^1(k_f, r_1'), d_{7,11} = -\frac{1}{c_f} g_n^2(k_f, r_1')$$

$$d_{8,1} = A_{s,0}^1(r_0'), d_{8,2} = D_{s,0}^1(r_0'), d_{8,10} = \Omega_{r_0} \frac{1}{c_{r_0}^2} g_n^1(k_f, r_0'), d_{8,11} = \Omega_{r_0} \frac{1}{c_{r_0}^2} g_n^2(k_f, r_0')$$

$$d_{9,7} = B_{s,0}^1(r_0'), d_{9,8} = E_{s,0}^1(r_0')$$

$$d_{9,6} = \Gamma_{s,0}^1(r_0')$$

$$d_{10,8} = \Gamma_{s,0}^1(r_0')$$

$$d_{11,7} = \frac{g_s(k_s, r_0')}{k_{s_0} r_0'}, d_{11,9} = \frac{g_s(k_s, r_0')}{k_{s_0} r_0'}, d_{11,10} = -\frac{1}{c_f} g_n^1(k_f, r_0')$$

$$d_{11,10} = -\frac{1}{c_f} g_n^1(k_f, r_0')$$
Case 4: Three Elastic Spheres (EEE - model)

In this case the vector $x$ and the non-zero elements of the matrix $D$ are

$$x = \begin{bmatrix} \alpha_{n,1} & \alpha_{s,2} & \beta_{n,1} & \beta_{s,2} & \gamma_{n,1} & \gamma_{s,2} & \alpha_{n,2} & \alpha_{s,2} & \beta_{n,1} & \beta_{s,2} & \gamma_{n,1} & \gamma_{s,2} & \alpha_{n,3} & \beta_{n,3} & \gamma_{n,3} & \gamma_{s,3} \end{bmatrix}^T$$

$$d_{1,1} = A_{n,2}(r_2^2), d_{1,2} = A_{s,2}(r_2^2), d_{1,5} = D_{n,2}(r_2^2), d_{1,6} = D_{n,2}(r_2^2)$$

$$d_{2,1} = E_{n,2}(r_2^2), d_{2,2} = E_{s,2}(r_2^2), d_{2,4} = E_{n,2}(r_2^2)$$

$$d_{3,1} = B_{n,2}(r_2^2), d_{3,2} = B_{s,2}(r_2^2), d_{3,5} = E_{n,2}(r_2^2), d_{3,6} = E_{n,2}(r_2^2)$$

$$d_{4,1} = \delta_n^1(\Omega r_1^1), d_{4,2} = \delta_s^1(\Omega r_1^1)$$

$$d_{4,3} = n(n+1)g_n^1(\Omega \sqrt{\frac{2+\frac{\lambda_2^{\prime}}{\mu_2^{\prime}}} \frac{1}{\Omega \sqrt{2+\frac{\lambda_2^{\prime}}{\mu_2^{\prime}}} r_1^1}}$$

$$d_{4,5} = n(n+1)g_n^2(\Omega \sqrt{\frac{2+\frac{\lambda_2^{\prime}}{\mu_2^{\prime}}} \frac{1}{\Omega \sqrt{2+\frac{\lambda_2^{\prime}}{\mu_2^{\prime}}} r_1^1}}$$

$$d_{4,7} = -g_n^1(\xi_{n,21}^1, \Omega r_1^1), d_{4,8} = -g_s^1(\xi_{s,21}^1, \Omega r_1^1)$$

$$d_{4,11} = -n(n+1)g_n^1(\xi_{n,21}^1, \Omega \sqrt{\frac{2+\frac{\lambda_2^{\prime}}{\mu_2^{\prime}}} r_1^1}) \frac{1}{\xi_{n,21}^1 \Omega \sqrt{2+\frac{\lambda_2^{\prime}}{\mu_2^{\prime}}} r_1^1}}$$

$$d_{4,12} = -n(n+1)g_n^2(\xi_{n,21}^1, \Omega \sqrt{\frac{2+\frac{\lambda_2^{\prime}}{\mu_2^{\prime}}} r_1^1}) \frac{1}{\xi_{n,21}^1 \Omega \sqrt{2+\frac{\lambda_2^{\prime}}{\mu_2^{\prime}}} r_1^1}}$$

$$d_{5,3} = g_n^1(\Omega \sqrt{\frac{2+\frac{\lambda_2^{\prime}}{\mu_2^{\prime}}} r_1^1}), d_{5,4} = g_n^2(\Omega \sqrt{\frac{2+\lambda_2^{\prime}}{\mu_2^{\prime}}} r_1^1), d_{5,9} = -g_s^1(\xi_{n,21}^1, \Omega \sqrt{2+\frac{\lambda_2^{\prime}}{\mu_2^{\prime}}} r_1^1)$$

$$d_{5,10} = -g_s^2(\xi_{n,21}^1, \Omega \sqrt{2+\frac{\lambda_2^{\prime}}{\mu_2^{\prime}}} r_1^1)$$

$$d_{6,1} = \delta_n^1(\Omega r_1^1) \frac{1}{\Omega r_1^1}, d_{6,2} = \delta_s^2(\Omega r_1^1) \frac{1}{\Omega r_1^1}$$

$$d_{6,5} = g_n^1(\Omega \sqrt{\frac{2+\frac{\lambda_2^{\prime}}{\mu_2^{\prime}}} r_1^1}) + \frac{g_n^1(\Omega \sqrt{\frac{2+\frac{\lambda_2^{\prime}}{\mu_2^{\prime}}} r_1^1})}{\Omega \sqrt{2+\frac{\lambda_2^{\prime}}{\mu_2^{\prime}}} r_1^1}$$
\[ d_{6,5} = g_n^2 \left( \frac{\Omega}{2 + \frac{\lambda_2'}{\mu_2'}} \right) \frac{g_n^2 \left( \frac{2 + \frac{\lambda_2'}{\mu_2'}}{\Omega} \right)}{2 + \frac{\lambda_2'}{\mu_2'}} \]

\[ d_{6,7} = -\frac{g_n^2 (\xi_{\mu,2} \Omega r_0')}{\xi_{\mu,2} \Omega r_0'} \quad d_{6,8} = -\frac{g_n^2 (\xi_{\mu,2} \Omega r_0')}{\xi_{\mu,2} \Omega r_0'} \]

\[ d_{6,11} = \left[ \frac{\hat{g}_n^2(\xi_{\mu,2} \Omega 2 + \frac{\lambda_1'}{\mu_1'})}{\xi_{\mu,2} \Omega \left( 2 + \frac{\lambda_1'}{\mu_1'} \right)} \right] \]

\[ d_{6,12} = \left[ \frac{\hat{g}_n^2(\xi_{\mu,2} \Omega 2 + \frac{\lambda_1'}{\mu_1'})}{\xi_{\mu,2} \Omega \left( 2 + \frac{\lambda_1'}{\mu_1'} \right)} \right] \]

\[ d_{7,1} = A_{n,2}^1 (r_1') \quad d_{7,2} = A_{n,2}^2 (r_1') \quad d_{7,8} = D_{n,2}^1 (r_1') \quad d_{7,6} = D_{n,2}^2 (r_1') \quad d_{7,7} = -A_{n,1}^1 (r_1') \]

\[ d_{7,8} = -A_{n,1}^2 (r_1') \quad d_{7,11} = -D_{n,1}^1 (r_1') \quad d_{7,12} = -D_{n,1}^2 (r_1') \]

\[ d_{8,3} = \Gamma_{n,3}^1 (r_1') \quad d_{8,4} = \Gamma_{n,2}^2 (r_1') \quad d_{8,5} = \Gamma_{n,1}^1 (r_1') \quad d_{8,10} = \Gamma_{n,1}^2 (r_1') \]

\[ d_{9,1} = B_{n,3}^1 (r_1') \quad d_{9,2} = B_{n,2}^2 (r_1') \quad d_{9,5} = E_{n,1}^1 (r_1') \quad d_{9,6} = E_{n,2}^2 (r_1') \]

\[ d_{9,7} = -B_{n,1}^1 (r_1') \quad d_{9,8} = -B_{n,1}^2 (r_1') \quad d_{9,11} = -E_{n,1}^2 (r_1') \quad d_{9,12} = -E_{n,1}^2 (r_1') \]

\[ d_{10,7} = \hat{g}_n^1 (\xi_{\mu,2} \Omega r_0') \quad d_{10,8} = \hat{g}_n^2 (\xi_{\mu,2} \Omega r_0') \]

\[ d_{10,11} = n(n + 1) \hat{g}_n^1 (\xi_{\mu,2} \Omega \sqrt{2 + \frac{\lambda_1'}{\mu_1'}}) \frac{1}{\xi_{\mu,2} \Omega \sqrt{2 + \frac{\lambda_1'}{\mu_1'}}} \]

\[ d_{10,12} = n(n + 1) \hat{g}_n^2 (\xi_{\mu,2} \Omega \sqrt{2 + \frac{\lambda_1'}{\mu_1'}}) \frac{1}{\xi_{\mu,2} \Omega \sqrt{2 + \frac{\lambda_1'}{\mu_1'}}} \]

\[ d_{10,13} = -\hat{g}_n^1 (\xi_{\mu,2} \Omega r_0') \]

-24-
\[
d_{10,15} = -n(n+1) \frac{g^i_n(\xi_{p,20}, \Omega, \sqrt{2 + \frac{\lambda^i_1}{\mu_0} r_0})}{\xi_{p,20} \Omega, \sqrt{2 + \frac{\lambda^i_0}{\mu_0} r_0}} \\
\]

\[
d_{11,0} = g^i_n(\xi_{p,20}, \Omega, \sqrt{2 + \frac{\lambda^i_1}{\mu_1} r_0}), \quad d_{11,10} = g^i_n(\xi_{p,20}, \Omega, \sqrt{2 + \frac{\lambda^i_1}{\mu_1} r_0}) \\
\]

\[
d_{11,14} = -g^i_n(\xi_{p,20}, \Omega, \sqrt{2 + \frac{\lambda^i_0}{\mu_0} r_0}) \\
\]

\[
d_{12,7} = g^i_n(\xi_{p,20}, \Omega, r_0) \frac{1}{\xi_{p,20} \Omega r_0}, \quad d_{12,8} = g^i_n(\xi_{p,21}, \Omega, r_0) \frac{1}{\xi_{p,21} \Omega r_0} \\
\]

\[
d_{12,11} = \frac{g^i_n(\xi_{p,21}, \Omega, \sqrt{2 + \frac{\lambda^i_1}{\mu_1} r_0})}{\xi_{p,21} \Omega \sqrt{2 + \frac{\lambda^i_1}{\mu_1} r_0}} + \frac{g^i_n(\xi_{p,21}, \Omega, \sqrt{2 + \frac{\lambda^i_0}{\mu_0} r_0})}{\xi_{p,21} \Omega \sqrt{2 + \frac{\lambda^i_0}{\mu_0} r_0}} \\
\]

\[
d_{12,12} = \frac{g^i_n(\xi_{p,21}, \Omega, \sqrt{2 + \frac{\lambda^i_1}{\mu_1} r_0})}{\xi_{p,21} \Omega \sqrt{2 + \frac{\lambda^i_1}{\mu_1} r_0}} + \frac{g^i_n(\xi_{p,21}, \Omega, \sqrt{2 + \frac{\lambda^i_0}{\mu_0} r_0})}{\xi_{p,21} \Omega \sqrt{2 + \frac{\lambda^i_0}{\mu_0} r_0}} \\
\]

\[
d_{12,13} = -g^i_n(\xi_{p,21}, \Omega, r_0) \frac{1}{\xi_{p,21} \Omega r_0} \\
\]

\[
d_{12,15} = \left[ \frac{g^i_n(\xi_{p,20}, \Omega, \sqrt{2 + \frac{\lambda^i_0}{\mu_0} r_0})}{\xi_{p,20} \Omega \sqrt{2 + \frac{\lambda^i_0}{\mu_0} r_0}} + \frac{g^i_n(\xi_{p,21}, \Omega, \sqrt{2 + \frac{\lambda^i_0}{\mu_0} r_0})}{\xi_{p,21} \Omega \sqrt{2 + \frac{\lambda^i_0}{\mu_0} r_0}} \right] \\
\]

\[
d_{13,7} = A^1_{n,1}(r_0), \quad d_{13,8} = A^2_{n,1}(r_0), \quad d_{13,11} = D^1_{n,1}(r_0), \quad d_{13,12} = D^2_{n,1}(r_0) \\
\]

\[
d_{13,13} = -A^1_{n,0}(r_0), \quad d_{13,15} = -D^1_{n,0}(r_0) \\
\]

\[
d_{14,9} = \Gamma^2_{n,1}(r_0), \quad d_{14,10} = \Gamma^2_{n,1}(r_0), \quad d_{14,14} = -\Gamma^2_{n,0}(r_0) \\
\]

\[
d_{15,7} = B^1_{n,1}(r_0), \quad d_{15,8} = B^2_{n,1}(r_0), \quad d_{15,11} = E^1_{n,1}(r_0) \\
\]

\[
d_{15,13} = E^2_{n,1}(r_0), \quad d_{15,13} = -B^1_{n,0}(r_0), \quad d_{15,15} = -E^1_{n,0}(r_0) \\
\]

where \( \xi_{p,21} = \frac{c_{pl}}{c_{p1}} \).