Ramsey Cardinals and the HNN Embedding Theorem

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In memory of Greg Hjorth

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The HNN Embedding Theorem

Theorem (Higman-Neumann-Neumann 1949)

If G is a countable group, then G can be embedded into a 2-generator group K_G.

Notation

- $\mathcal{G}$ denotes the Polish space of countably infinite groups.
- $\mathcal{G}_{fg}$ denotes the Polish space of finitely generated groups.

Theorem

There does not exist a Borel map $G \mapsto K_G$ from $\mathcal{G}$ to $\mathcal{G}_{fg}$ such that for all $G, H \in \mathcal{G}$,

- *$G \hookrightarrow K_G$; and*
- *if $G \cong H$, then $K_G \cong K_H$.*

Acknowledging the existence of large cardinals ...

Main Theorem (*LC*)

- *Suppose that $G \mapsto K_G$ is any Borel map from $\mathcal{G}$ to $\mathcal{G}_{fg}$ such that $G \hookrightarrow K_G$ for all $G \in \mathcal{G}$.*
- *Then there exists an uncountable Borel family $\mathcal{F} \subseteq \mathcal{G}$ of pairwise isomorphic groups such that the groups $\{ K_G \mid G \in \mathcal{F} \}$ are pairwise incomparable with respect to relative constructibility; i.e., if $G \neq H \in \mathcal{F}$, then $K_G \notin L[K_H]$ and $K_H \notin L[K_G]$.*

Remarks

- (*LC*): There exists a Ramsey cardinal κ.
- In *ZFC*, we can find an uncountable Borel family $\mathcal{F}$ such that the groups $\{ K_G \mid G \in \mathcal{F} \}$ are pairwise incomparable with respect to embeddability.

Countable Quasi-orders

Definition

The relation $\preceq$ on the Polish space X is a countable quasi-order if:

(a) *$\preceq$ is reflexive and transitive.*

(b) *For all $x \in X$, the set $\{ y \in X \mid y \preceq x \}$ is countable.*

Some countable Borel quasi-orders

- The embeddability relation on $\mathcal{G}_{fg}$.
- The Turing reducibility relation $\leq_T$ on $2^{\mathbb{N}}$.

A countable Σ^1_2 quasi-order (LC)

The relative constructibility relation $\leq_c$ on $2^{\mathbb{N}}$ defined by

$$x \leq_c y \iff x \in L[y].$$

Towards a proof of the Main Theorem ...

Question

What is known about the kernels of homomorphisms from complete analytic equivalence relations to countable Borel equivalence relations?

Answer (Kechris)

Not a lot!

Definition

- $\mathrm{Inj}(\mathbb{N}, 2^{\mathbb{N}})$ *is the Polish space of all injective maps* $z : \mathbb{N} \to 2^{\mathbb{N}}$.
- E_{cntble} *is the Borel equivalence relation on* $\mathrm{Inj}(\mathbb{N}, 2^{\mathbb{N}})$ *defined by*

$$z \, E_{cntble} \, z' \iff \{ z(n) \mid n \in \mathbb{N} \} = \{ z'(n) \mid n \in \mathbb{N} \}.$$

The Main Lemma

Main Lemma

Suppose that X is a Polish space and that $\theta : \text{Inj}(\mathbb{N}, 2^{\mathbb{N}}) \to X$ is any Borel map. Then at least one of the following must hold:

(a) *There exists $x \in X$ such that for all $r \in 2^{\mathbb{N}}$, there exists $z \in \text{Inj}(\mathbb{N}, 2^{\mathbb{N}})$ with $r \in \text{range}(z)$ such that $\theta(z) = x$.*

(b) *For each countable Borel quasi-order $\preccurlyeq$ on X, there exists a perfect subset $P \subseteq \text{Inj}(\mathbb{N}, 2^{\mathbb{N}})$ such that*

   (i) *$y \; E_{cntble} \; z$ for all $y, z \in P$; and*

   (ii) *$\theta(y), \theta(z)$ are incomparable with respect to $\preccurlyeq$ for all $y \neq z \in P$.*

Moreover, if (LC) holds, then the conclusion also holds with respect to the quasi-order $\leq_c$ of relative constructibility.

The Proof of the Main Theorem

- Suppose that $\varphi : \mathcal{G} \to \mathcal{G}_{fg}$ is a Borel map such that $G \hookrightarrow \varphi(G)$ for all $G \in \mathcal{G}$.
- Let $\{ H_r \mid r \in 2^{\mathbb{N}} \} \subseteq \mathcal{G}$ be a Borel family of pairwise nonisomorphic 2-generator groups. (B. H. Neumann 1937)
- Let $\psi : \mathrm{Inj}(\mathbb{N}, 2^{\mathbb{N}}) \to \mathcal{G}$ be the injective Borel map defined by

$$\psi(z) = H_{z(0)} \times H_{z(1)} \times \cdots \times H_{z(n)} \times \cdots$$

  and consider $\theta = \varphi \circ \psi : \mathrm{Inj}(\mathbb{N}, 2^{\mathbb{N}}) \to \mathcal{G}_{fg}$.
- First suppose that there exists a group $G \in \mathcal{G}_{fg}$ such that for all $r \in 2^{\mathbb{N}}$, there exists $z \in \mathrm{Inj}(\mathbb{N}, 2^{\mathbb{N}})$ such that $r \in \mathrm{range}(z)$ and $\theta(z) = G$.
- Then H_r embeds into G for all $r \in 2^{\mathbb{N}}$, which is impossible since G has only countably many 2-generator subgroups!

The Proof of the Main Theorem

- Let $\preceq$ be either the embeddability relation or the relative constructibility relation on $\mathcal{G}_{fg}$.
- Then there exists a perfect subset $P \subseteq \text{Inj}(\mathbb{N}, 2^{\mathbb{N}})$ such that
  (i) $y \, E_{cntble} \, z$ for all $y, z \in P$; and
  (ii) $\theta(y), \theta(z)$ are incomparable with respect to $\preccurlyeq$ for all $y \neq z \in P$.
- Hence $\mathcal{F} = \psi(P) \subseteq \mathcal{G}$ is an uncountable Borel family of pairwise isomorphic groups such that the groups $\{ \varphi(G) \mid G \in \mathcal{F} \}$ are pairwise incomparable with respect to $\preceq$.

Towards a proof of the Main Lemma ...

Notation

- From now on, we work within a fixed set-theoretic universe V.
- Let $\mathbb{P}$ be a forcing notion.

Definition

- *The relation R on the Polish space X is $\mathbf{\Sigma}^1_n$ if $R(\bar{v})$ has the form*

$$(\exists x_1 \in X_1)(\forall x_2 \in X_2) \cdots B(x_1, x_2, \cdots, \bar{v}),$$

  where $X_1, \cdots, X_n$ are Polish spaces and $B(\bar{x}, \bar{v})$ is a Borel relation.
- *In this case, $R^{V^{\mathbb{P}}}$ denotes the relation obtained by applying the definition of R within the generic extension $V^{\mathbb{P}}$.*
- *R is absolute for $V^{\mathbb{P}}$ if $R^{V^{\mathbb{P}}} \cap V = R$.*

Shoenfield Absoluteness

Theorem (Shoenfield)

If $R \in V$ is a $\mathbf{\Sigma}^1_2$ relation, then R is absolute for every generic extension $V^{\mathbb{P}}$.

An Application

If $\preceq$ is a countable Borel quasi-order on the Polish space X, then $\preceq^{V^{\mathbb{P}}}$ is a countable Borel quasi-order on $X^{V^{\mathbb{P}}}$.

Proof.

Let $\mathrm{Perf}(X)$ be the Polish space of nonempty perfect subsets of X. Then $\preceq$ is countable if and only if

$$(\forall x \in X)\,(\forall P \in \mathrm{Perf}(X))\,(\exists y \in X)\,[\, y \in P \wedge y \not\preceq x \,].$$

□

Martin-Solovay Absoluteness

Theorem (Martin-Solovay)

Suppose that κ is a Ramsey cardinal. If $R \in V$ is a $\mathbf{\Sigma}^1_3$ relation and $|\mathbb{P}| < \kappa$, then R is absolute for $V^{\mathbb{P}}$.

An Application (*LC*)

$\leq_c$ is a countable $\mathbf{\Sigma}^1_2$ quasi-order on $2^{\mathbb{N}}$.

Proof.

If $\mathbb{P}$ is the poset of finite functions $p : \omega \to \omega_1$, then for all $x \in 2^{\mathbb{N}} \cap V$,

$$V^{\mathbb{P}} \models (\exists f \in (2^{\mathbb{N}})^{\mathbb{N}})(\forall z \in 2^{\mathbb{N}})[\, z \in L[x] \implies (\exists n)\ f(n) = z\,].$$

By Martin-Solovay, this $\mathbf{\Sigma}^1_3(x)$ statement also holds in V. □

Virtual equivalence classes

Definition (Kanovei après Hjorth)

Let E be a Borel equivalence relation on the Polish space X and let $\mathbb{P}$ be a forcing notion. Then a $\mathbb{P}$-name τ is a virtual E-class if:

- $\Vdash_{\mathbb{P}} \ \tau \in X^{V^{\mathbb{P}}}$
- $\Vdash_{\mathbb{P}\times\mathbb{P}} \ \tau_{\mathit{left}} \ E^{V^{\mathbb{P}\times\mathbb{P}}} \ \tau_{\mathit{right}}$

Here τ_{left}, τ_{right} are the $(\mathbb{P}\times\mathbb{P})$-names such that if $G \times H$ is $(\mathbb{P}\times\mathbb{P})$-generic, then $\tau_{\mathit{left}}[G\times H] = \tau[G]$ and $\tau_{\mathit{right}}[G\times H] = \tau[H]$.

Virtual equivalence classes

Example

- Let $E = E_{cntble}$ and let $\mathbb{P}$ consist of all finite injective partial functions $p : \mathbb{N} \to 2^{\mathbb{N}}$.
- If G is $\mathbb{P}$-generic, then $g = \bigcup G$ is a bijection between $\mathbb{N}$ and $2^{\mathbb{N}} \cap V$.
- Hence if τ is the canonical $\mathbb{P}$-name such that $\tau[G] = g$, then τ is a virtual E_{cntble}-class.

A reminder ...

Main Lemma

Suppose that X is a Polish space and that $\theta : \mathrm{Inj}(\mathbb{N}, 2^{\mathbb{N}}) \to X$ is any Borel map. Then at least one of the following must hold:

(a) *There exists $x \in X$ such that for all $r \in 2^{\mathbb{N}}$, there exists $z \in \mathrm{Inj}(\mathbb{N}, 2^{\mathbb{N}})$ with $r \in \mathrm{range}(z)$ such that $\theta(z) = x$.*

(b) *For each countable Borel quasi-order $\preccurlyeq$ on X, there exists a perfect subset $P \subseteq \mathrm{Inj}(\mathbb{N}, 2^{\mathbb{N}})$ such that*

  (i) *$y \, E_{cntble} \, z$ for all $y, z \in P$; and*

  (ii) *$\theta(y), \theta(z)$ are incomparable with respect to $\preccurlyeq$ for all $y \neq z \in P$.*

Moreover, if (LC) holds, then the conclusion also holds with respect to the quasi-order $\leq_c$ of relative constructibility.

Towards a proof of the Main Lemma ...

- Let $\theta : \mathrm{Inj}(\mathbb{N}, 2^{\mathbb{N}}) \to X$ be any Borel map.
- Let $\preceq$ be either a countable Borel quasi-order on X or else the relative constructibility relation $\leq_c$.

Notation

- $x \perp y \iff x, y$ are $\preceq$-incomparable.
- $x \,||\, y \iff x, y$ are $\preceq$-comparable.

- Let $\mathbb{P}$ consist of all finite injective partial functions $p : \mathbb{N} \to 2^{\mathbb{N}}$ and let τ be the corresponding virtual E_{cntble}-class.

The Fundamental Dichotomy

Are $\theta(\tau_{left}), \theta(\tau_{right})$ comparable with respect to $\preceq^{V^{\mathbb{P}\times\mathbb{P}}}$?

Case 1: $(\exists p_0 \in \mathbb{P})\ \langle p_0, p_0 \rangle \Vdash \theta(\tau_{\text{left}}) \mid\mid \theta(\tau_{\text{right}})$.

Claim

There exists $p_1 \leq p_0$ such that $\langle p_1, p_1 \rangle \Vdash \theta(\tau_{\text{left}}) = \theta(\tau_{\text{right}})$.

Proof.

- Suppose not and let $\mathbb{Q}$ collapse $\mathcal{P}(\mathbb{P} \times \mathbb{P})$ to a countable set.
- Working in $V^{\mathbb{Q}}$, there exists a perfect subset $P \subseteq \text{Inj}(\mathbb{N}, 2^{\mathbb{N}})$ such that $\theta(P)$ is an uncountable Borel set of pairwise $\preceq$-comparable elements.
- Let $Z \subseteq \theta(P)$ be a perfect subset.
- By Kuratowski-Ulam, both $A = \{ (x, y) \in Z \times Z \mid x \preceq y \}$ and $B = \{ (x, y) \in Z \times Z \mid y \preceq x \}$ are meager subsets of $Z \times Z$.
- Since $Z \times Z = A \cup B$, this contradicts the Baire Category Theorem.

□

Case 1: $(\exists p_0 \in \mathbb{P})\ \langle p_0, p_0 \rangle \Vdash \theta(\tau_{\text{left}}) \parallel \theta(\tau_{\text{right}})$.

Working in V and assuming that $X = [0,1]$, we can inductively define conditions

$$p_1 \geq p_2 \geq p_3 \geq \cdots \geq p_n \geq \cdots$$

and closed intervals $I_n \subseteq [0,1]$ with rational endpoints

$$I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$$

such that the following conditions hold:

- $|I_n| = 2^{-(n-1)}$
- $p_n \Vdash \theta(\tau) \in I_n$.

Still working in V, let

$$\bigcap_{n \geq 1} I_n = \{x\}.$$

Case 1: $(\exists p_0 \in \mathbb{P})\ \langle p_0, p_0 \rangle \Vdash \theta(\tau_{\text{left}}) \,||\, \theta(\tau_{\text{right}})$.

Claim

$p_1 \Vdash \theta(\tau) = x$.

Proof.

- Otherwise, there exists $q \leq p_1$ and $n \geq 1$ such that $q \Vdash \theta(\tau) \notin I_n$.
- But then $\langle q, p_n \rangle \leq \langle p_1, p_1 \rangle$ satisfies

$$\langle q, p_n \rangle \Vdash \theta(\tau_{\text{left}}) \notin I_n \ \text{ and } \ \theta(\tau_{\text{right}}) \in I_n,$$

which is a contradiction.

□

Case 1: $(\exists p_0 \in \mathbb{P})\ \langle p_0, p_0 \rangle \Vdash \theta(\tau_{\text{left}}) \mid\mid \theta(\tau_{\text{right}})$.

- Let $G \subseteq \mathbb{P}$ be V-generic with $p_1 \in G$.
- Then $V[G] \vDash \theta(\tau[G]) = x$.
- Hence for each $r \in 2^{\mathbb{N}} \cap V$,

$$V[G] \vDash (\exists z \in \text{Inj}(\mathbb{N}, 2^{\mathbb{N}}))(\exists n \in \mathbb{N})\ [z(n) = r \text{ and } \theta(z) = x].$$

- By Shoenfield Absoluteness, this $\mathbf{\Sigma}^1_1$ property of the reals $r, x \in 2^{\mathbb{N}} \cap V$ must also hold in V.
- Thus, in V, for all $r \in 2^{\mathbb{N}}$, there exists $z \in \text{Inj}(\mathbb{N}, 2^{\mathbb{N}})$ with $r \in \text{range}(z)$ such that $\theta(z) = x$.

Case 2: $(\forall p \in \mathbb{P})\ \langle p, p \rangle \not\Vdash \theta(\tau_{\text{left}}) \,||\, \theta(\tau_{\text{right}})$.

- Once again, let $\mathbb{Q}$ collapse $\mathcal{P}(\mathbb{P} \times \mathbb{P})$ to a countable set.
- Then $V^{\mathbb{Q}}$ satisfies the following statement:

$$(\exists P \in \text{Perf}(\text{Inj}(\mathbb{N}, 2^{\mathbb{N}})))\,(\forall x)\,(\forall y)$$
$$[\,(x, y \in P \wedge x \neq y) \implies (x\, E_{cntble}\, y \wedge \theta(x) \perp \theta(y))\,].$$

- Applying either Shoenfield or Martin-Solovay Absoluteness, this statement also holds in V.
- This completes the proof of the Main Lemma.

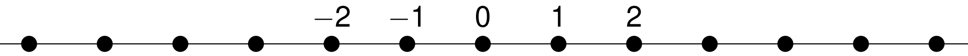
Cayley graphs of finitely generated groups

Definition

Let G be a f.g. group and let $S \subseteq G \smallsetminus \{1_G\}$ be a finite generating set. Then the *Cayley graph* $\mathrm{Cay}(G, S)$ *is the graph with vertex set G and edge set*

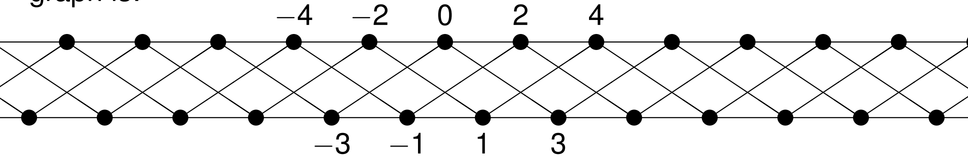
$$E = \{\{x, y\} \mid y = xs \text{ for some } s \in S \cup S^{-1}\}.$$

For example, when $G = \mathbb{Z}$ and $S = \{1\}$, then the corresponding Cayley graph is:



But which Cayley graph?

However, when $G = \mathbb{Z}$ and $S = \{2, 3\}$, then the corresponding Cayley graph is:



Theorem

There does not exist an Borel choice of generators for each f.g. group which has the property that isomorphic groups are assigned isomorphic Cayley graphs.

Sketch proof.

Apply some basic geometric group theory and ergodic theory. □

Just infinite groups

Definition

An infinite group G is said to be just infinite if every proper quotient of G is finite.

Some Examples

- Infinite simple groups are just infinite.
- $SL_3(\mathbb{Z})$ is just infinite.

Remark

An interesting theory of just infinite groups has been developed by Girgorchuk, Wilson, etc.

Just infinite groups

Proposition

Every infinite f.g. group G has a just infinite quotient G/N.

Proof.

- It is enough to show that the partially ordered set

$$\mathcal{N} = \{ N \trianglelefteq G \mid G/N \text{ is infinite } \}$$

  has a maximal element.
- Suppose that $N_0 \leqslant \cdots \leqslant N_\ell \leqslant \cdots$ is a chain and let $N = \bigcup N_\ell$.
- If $N \notin \mathcal{N}$, then $[\, G : N \,] < \infty$ and this implies that N is f.g., which is a contradiction.

$\square$

Just infinite groups

Theorem

There does not exist a Borel map $G \mapsto Q_G$ from $\mathcal{G}_{fg}$ to $\mathcal{G}_{fg}$ such that for all $G, H \in \mathcal{G}_{fg}$,

- *Q_G is a just infinite quotient of G; and*
- *if $G \cong H$, then $Q_G \cong Q_H$.*

Sketch proof.

Apply some not so basic topological dynamics. □

Question

Is there an inevitable non-uniformity in the proofs in this area?